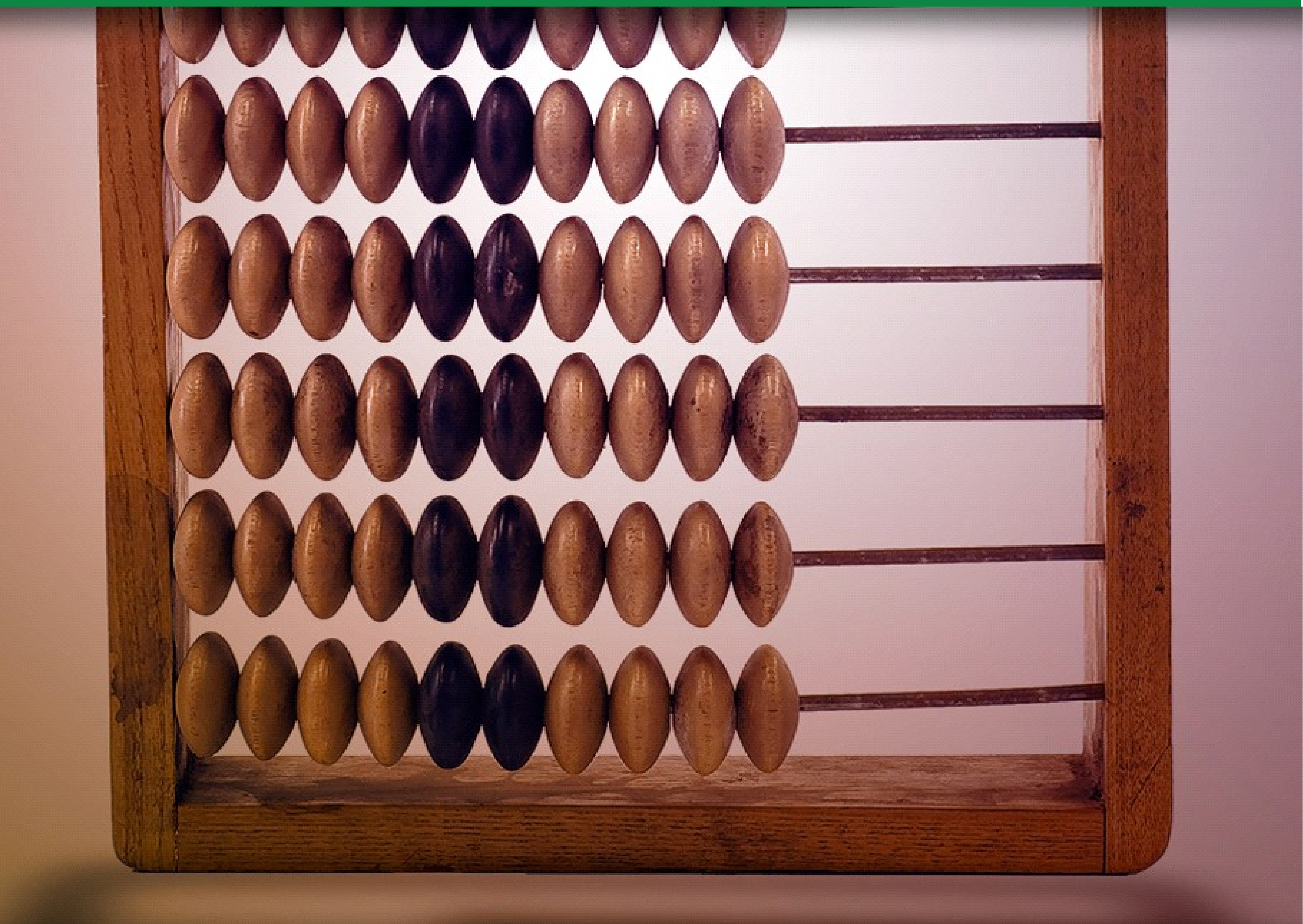
LEIF MEJLBRO

REAL FUNCTIONS OF SEVERAL VARIABLES – SPACE INT...



Leif Mejlbro

Real Functions of Several Variables

Examples of Space Integrals

Calculus 2c-6

Real Functions of Several Variables – Examples of Space Integrals – Calculus 2c-6

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Contents

Preface

In this volume I present some examples of *space integrals*, cf. also *Calculus 2b, Functions of Several Variables*. Since my aim also has been to demonstrate some solution strategy I have as far as possible structured the examples according to the following form

A *Awareness*, i.e. a short description of what is the problem.

D *Decision*, i.e. a reflection over what should be done with the problem.

I *Implementation*, i.e. where all the calculations are made.

C *Control*, i.e. a test of the result.

This is an ideal form of a general procedure of solution. It can be used in any situation and it is not linked to Mathematics alone. I learned it many years ago in the Theory of Telecommunication in a situation which did not contain Mathematics at all. The student is recommended to use it also in other disciplines.

One is used to from high school immediately to proceed to **I**. *Implementation*. However, examples and problems at university level are often so complicated that it in general will be a good investment also to spend some time on the first two points above in order to be absolutely certain of what to do in a particular case. Note that the first three points, **ADI**, can always be performed.

This is unfortunately not the case with **C** *Control*, because it from now on may be difficult, if possible, to check one's solution. It is only an extra securing whenever it is possible, but we cannot include it always in our solution form above.

I shall on purpose not use the logical signs. These should in general be avoided in Calculus as a shorthand, because they are often (too often, I would say) misused. Instead of $\wedge$ I shall either write "and", or a comma, and instead of $\vee$ I shall write "or". The arrows $\Rightarrow$ and $\Leftrightarrow$ are in particular misunderstood by the students, so they should be totally avoided. Instead, write in a plain language what you mean or want to do.

It is my hope that these examples, of which many are treated in more ways to show that the solutions procedures are not unique, may be of some inspiration for the students who have just started their studies at the universities.

Finally, even if I have tried to write as careful as possible, I doubt that all errors have been removed. I hope that the reader will forgive me the unavoidable errors.

Leif Mejlbro
13th October 2007

1 Space integral, rectangular coordinates

Example 1.1 *Compute in each of the following cases the given space integral over a point set*

$$A = \{(x,y,z) \mid (x,y) \in B, \quad Z_1(x,y) \le z \le Z_2(x,y)\}.$$

1) *The space integral* $\int_A xy^2z\,d\Omega$*, where the plane point set* B *is given by* $x \ge 0$, $y \ge 0$ *and* $x+y \le 1$*, and where* $Z_1(x,y) = 0$ *and* $Z_2(x,y) = 2-x-y$.

2) *The space integral* $\int_A xy^2z^3\,d\Omega$*, where the plane point set* B *is given by* $0 \le x \le y \le 1$*, and where* $Z_1(x,y) = 0$ *and* $Z_2(x,y) = xy$.

3) *The space integral* $\int_A z\,d\Omega$*, where the plane point set* B *is given by* $0 \le x \le 6$ *and* $2-x \le y \le 3-\dfrac{x}{2}$*, and where* $Z_1(x,y) = 0$ *and* $Z_2 = \sqrt{16-y^2}$.

4) *The space integral* $\int_A y\,d\Omega$*, where the plane point set* B *is given by* $-2 \le y \le 1$ *and* $y^2 \le x \le 2-y$*, and where* $Z_1(x,y) = 0$ *and* $Z_2(x,y) = 4-2x-2y$.

5) *The space integral* $\int_A \dfrac{1}{x^2y^2z^2}\,d\Omega$*, where the plane point set* B *is given by* $1 \le x \le \sqrt{3}$ *and* $\dfrac{1}{1+x^2} \le y \le 1$*, and where* $Z_1(x,y) = \dfrac{1}{1+x^2}$ *and* $Z_2(x,y) = 1+x^2$.

6) *The space integral* $\int_A yz\,d\Omega$*, where the plane point set* B *is given by* $0 \le x \le 1$ *and* $0 \le y \le x$*, and where* $Z_1(x,y) = 0$ *and* $Z_2(x,y) = 2-2x$.

   [Cf. **Example 1.2.6**.]

7) *The space integral* $\int_A xz\,d\Omega$*, where the plane point set* B *is given by* $0 \le x \le 1$ *and* $0 \le y \le 1$*, and where* $Z_1(x,y) = 0$ *and* $Z_2(x,y) = 1-y$.

   [Cf. **Example 1.2.7**.]

8) *The space integral* $\int_A z\,d\Omega$*, where the plane point set* B *is given by* $\sqrt{x^2+y^2} \le 2$*, and where* $Z_1(x,y) = 0$ *and* $Z_2(x,y) = 2-\sqrt{x^2+y^2}$.

   [Cf. **Example 1.2.8**]

A Space integral in rectangular coordinates.

D Apply the theorem of reductions.

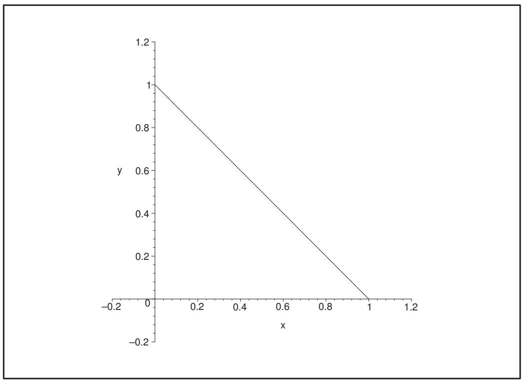

Figure 1: The domain B of **Example 1.1.1**.

I 1) By the theorem of reduction,

$$\begin{aligned}
\int_A xy^2 z\, d\Omega &= \int_B xy^2 \left\{ \int_0^{2-x-y} z\, dz \right\} dx dy = \frac{1}{2}\int_B x^2 y (2-x-y)^2\, dx dy \\
&= \frac{1}{2}\int_B x^2 y \left\{ (2-x)^2 - 2(2-x)y + y^2 \right\} dx dy \\
&= \frac{1}{2}\int_0^1 x^2 \left\{ \int_0^{1-x} \left[(2-x)^2 y - 2(2-x)y^2 + y^3 \right] dy \right\} dx \\
&= \frac{1}{2}\int_0^1 x^2 \left[\frac{1}{2}(2-x)^2 y^2 - \frac{2}{3}(2-x)y^3 + \frac{1}{4}\, y^4 \right]_{y=0}^{1-x} dx \\
&= \frac{1}{24}\int_0^1 x^2 \left\{ 6(2-x)^2(1-x)^2 - 8(2-x)(1-x)^3 + 3(1-x)^4 \right\} dx \\
&= \frac{1}{24}\int_0^1 x^2(1-x)^2 \left\{ 6(4-4x+x^2) - 8(2-3x+x^2) + 3(1-2x+x^2) \right\} dx \\
&= \frac{1}{24}\int_0^1 (x^4 - 2x^3 + x^2)(x^2 - 6x + 11)\, dx \\
&= \frac{1}{24}\int_0^1 \left\{ x^6 - 8x^5 + 24x^4 - 28x^3 + 11x^2 \right\} dx \\
&= \frac{1}{24}\left[\frac{1}{7}\, x^7 - \frac{4}{3}\, x^6 + \frac{24}{5}\, x^5 - 7x^4 + \frac{11}{3}\, x^3 \right]_0^1 \\
&= \frac{1}{24}\left(\frac{1}{7} - \frac{4}{3} + \frac{24}{5} - 7 + \frac{11}{3} \right) = \frac{1}{24}\left(\frac{1}{7} + 2 + \frac{1}{3} + 5 - \frac{1}{5} - 7 \right) \\
&= \frac{1}{24}\left(\frac{1}{3} - \frac{1}{5} + \frac{1}{7} \right) = \frac{1}{24}\left(\frac{2}{15} + \frac{1}{7} \right) = \frac{29}{24 \cdot 105} = \frac{29}{2520}.
\end{aligned}$$

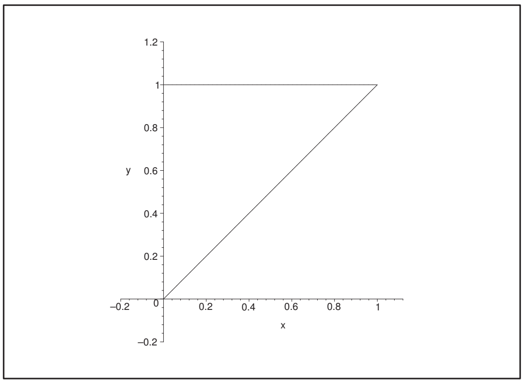

Figure 2: The domain B of **Eksempel 1.1.2**.

2) By the theorem of reduction,

$$\begin{aligned}\int_A xy^2z^3\,d\Omega &= \int_B xy^2\left\{\int_0^{xy} z^3\,dz\right\}dxdy = \frac{1}{4}\int_B xy\left[z^4\right]_{z=0}^{xy}dxdy = \frac{1}{4}\int_B x^5y^6\,dxdy \\ &= \frac{1}{4}\int_0^1 y^6\left\{\int_0^y x^5\,dx\right\}dy = \frac{1}{24}\int_0^1 y^{12}\,dy = \frac{1}{24\cdot 13} = \frac{1}{312}.\end{aligned}$$

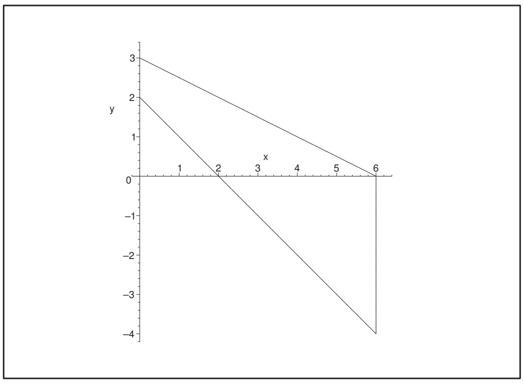

Figure 3: The domain B of **Example 1.1.3**.

3) By the theorem of reduction,

$$\begin{aligned}
\int_A z\,d\Omega &= \int_B \left\{\int_0^{\sqrt{16-y^2}} z\,dz\right\} dxdy = \frac{1}{2}\int_B (16-y^2)\,dxdy \\
&= \frac{1}{2}\int_0^6 \left\{\int_{2-x}^{3-\frac{x}{2}} (16-y^2)\,dy\right\} dx = \frac{1}{2}\int_0^6 \left[16y-\frac{1}{3}\,y^3\right]_{y=2-x}^{3-\frac{x}{2}} dx \\
&= \frac{1}{2}\int_0^6 \left\{16\left(3-\frac{x}{2}\right)-\frac{1}{3}\left(3-\frac{x}{2}\right)^3-16(2-x)+\frac{1}{3}(2-x)^3\right\} dx \\
&= \frac{1}{2}\int_0^6 \left\{16+8x+\frac{1}{24}\,(x-6)^2-\frac{1}{3}\,(x-2)^3\right\} dx \\
&= \frac{1}{2}\left[16x+4x^2+\frac{1}{96}\,(x-6)^4-\frac{1}{12}\,(x-2)^4\right]_0^6 \\
&= \frac{1}{2}\left\{96+144+0-\frac{4^4}{12}-\frac{6^4}{96}+\frac{2^4}{12}\right\} = \frac{1}{2}\left\{240-\frac{64}{3}-\frac{216}{16}+\frac{4}{3}\right\} \\
&= \frac{1}{2}\left\{220-\frac{27}{2}\right\} = \frac{413}{4}.
\end{aligned}$$

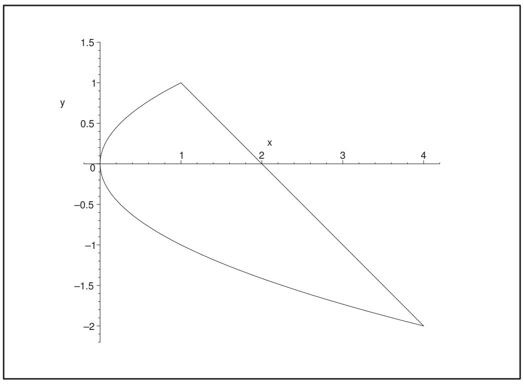

Figure 4: The domain B of **Example 1.1.4**.

4) By the theorem of reduction,

$$\begin{aligned}
\int_A y\,d\Omega &= \int_B y\left\{\int_0^{4-2x-2y} dz\right\} dx dy = \int_B y(4-2x-2y)\,dx dy \\
&= \int_{-2}^1 y\left\{\int_{y^2}^{2-y}(4-2x-2y)\,dx\right\} dy = \int_{-2}^1 y\left[4x-x^2-2xy\right]_{x=y^2}^{2-y} dy \\
&= \int_{-2}^1 y\left\{4(2-y)-(2-y)^2-2y(2-y)-4y^2+y^4+2y^3\right\} dy \\
&= \int_{-2}^1 y\left\{8-4y-4+4y-y^2-4y+2y^2-4y^2+2y^3+y^4\right\} dy \\
&= \int_{-2}^1 (y^5+2y^4-3y^3-4y^2+4y)\,dy = \left[\frac{1}{6}\,y^6+\frac{2}{5}\,y^5-\frac{3}{4}\,y^4-\frac{4}{3}\,y^3+2y^2\right]_{-2}^1 \\
&= \frac{1}{6}+\frac{2}{5}-\frac{3}{4}-\frac{4}{3}+2-\frac{2^6}{6}+\frac{2}{5}\cdot 2^5+\frac{3}{4}\cdot 2^4-\frac{4}{3}\cdot 2^3-8=-\frac{81}{20}.
\end{aligned}$$

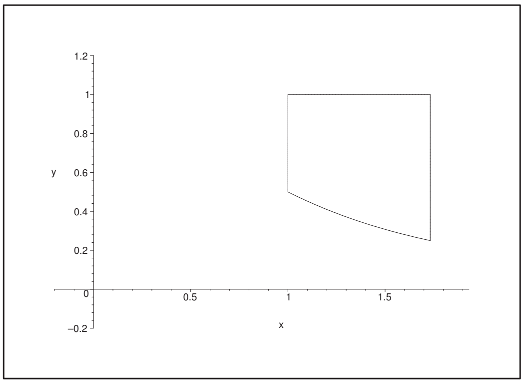

Figure 5: The domain B of **Example 1.1.5**.

5) First note that the z-integral does not depend on y. By exploiting this observation we get by the theorem of reduction,

$$\begin{aligned}
\int_A \frac{1}{x^2y^2z^2}\,d\Omega &= \int_1^{\sqrt{3}} \frac{1}{x^2}\left\{\int_{\frac{1}{1+x^2}}^1 \frac{1}{y^2}\left(\int_{\frac{1}{1+x^2}}^{1+x^2} \frac{1}{z^2}\,dz\right)dy\right\}dx \\
&= \int_1^{\sqrt{3}} \frac{1}{x^2}\left[-\frac{1}{y}\right]_{\frac{1}{1+x^2}}^{1} \cdot \left[-\frac{1}{z}\right]_{\frac{1}{1+x^2}}^{1+x^2} dx \\
&= \int_1^{\sqrt{3}} \frac{1}{x^2}\left(1+x^2-1\right)\cdot\left(1+x^2-\frac{1}{1+x^2}\right)dx \\
&= \int_1^{\sqrt{3}} \left(1+x^2-\frac{1}{1+x^2}\right)dx = \left[\frac{x^3}{3}+x-\ \text{Arctan}\ x\right]_1^{\sqrt{3}} \\
&= \frac{3\sqrt{3}}{3}+\sqrt{3}-\ \text{Arctan}\ \sqrt{3}-\frac{1}{3}-1+\ \text{Arctan}\ 1 \\
&= 2\sqrt{3}-\frac{4}{3}-\frac{\pi}{3}+\frac{\pi}{4} = 2\sqrt{3}-\frac{4}{3}-\frac{\pi}{12}.
\end{aligned}$$

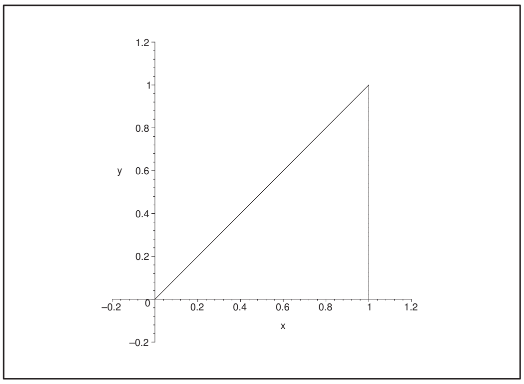

Figure 6: The domain B of **Example 1.1.6**.

6) By the theorem of reduction,

$$\begin{aligned}\int_A yz\,d\Omega &= \int_0^1\left\{\int_0^x y\left(\int_0^{2-2x} z\,dz\right)dy\right\}dx = \int_0^1\left[\frac{y^2}{2}\right]_0^x\cdot\left[\frac{z^2}{2}\right]_0^{2-2x}dx\\ &= \frac{1}{4}\int_0^1 x^2\cdot(2-2x)^2\,dx = \int_0^1 x^2(1-x^2)\,dx = \int_0^1 x^2(x^2-2x+1)\,dx\\ &= \int_0^1 (x^4-2x^3+x^2)\,dx = \frac{1}{5}-\frac{2}{4}+\frac{1}{3} = \frac{6-15+10}{30} = \frac{1}{30}.\end{aligned}$$

REMARK. The domain is also described by

$$0\le z\le 2,\qquad 0\le y\le x\le 1-\frac{z}{2},$$

cf. **Example 1.2.6**. The two examples therefore give the same result. ◊

7) Here, $B = [0,1] \times [0,1]$, thus it follows by the theorem of reduction that

$$\int_A xz\,d\Omega = \int_0^1 x\,dx \cdot \int_0^1 \left\{ \int_0^{1-y} z\,dz \right\} dy = \frac{1}{2} \int_0^1 \frac{1}{2}\,(1-y)^2\,dy = \frac{1}{4} \int_0^1 t^2\,dt = \frac{1}{12}.$$

8) Here, B is the closed disc of centrum $(0,0)$ and radius 2. By using the theorem of reduction in semi-polar coordinates,

$$\begin{aligned}\int_A z\,d\Omega &= 2\pi \int_0^2 \left\{ \int_0^{2-\varrho} z\,dz \right\} \varrho\,d\varrho = \pi \int_0^2 (2-\varrho)^2 \varrho\,d\varrho = \pi \int_0^2 (\varrho^3 - 4\varrho^2 + 4\varrho)\,d\varrho \\ &= \pi \left[\frac{\varrho^4}{4} - \frac{4}{3}\,\varrho^3 + 2\varrho^2 \right]_0^2 = \pi \left\{ 4 - \frac{32}{3} + 8 \right\} = \frac{4\pi}{3}.\end{aligned}$$

Example 1.2 *Calculate in each of the following cases the given space integral over a point set*

$A = \{(x,y,z) \mid \alpha \le z \le \beta, \quad (x,y) \in B(z)\}$.

1) The space integral $\int_A z^2\,d\Omega$, *where* $B(z)$ *is given by* $|x| \le z$ *and* $|y| \le 2z$ *for* $z \in [0,1]$.

2) The space integral $\int_A xz\,d\Omega$, *where* $B(z)$ *is given by* $0 \le x$, $0 \le y$ *and* $x + y \le z^2$ *for* $z \in [0,1]$.

3) The space integral $\int_A xy^2 z\,d\Omega$, *where* $B(z)$ *is given by* $0 \le x$, $0 \le y$ *and* $x + y \le 2 - z$ *for* $z \in [1,2]$.

4) The space integral $\int_A \frac{1}{xy^2}\,d\Omega$, *where* $B(z)$ *is given by* $1 \le x \le z$ *and* $z \le y \le z$ *for* $z \in [1,3]$.

5) The space integral $\int_A \left(\frac{\sin z}{z} \right)^2 d\Omega$, *where* $B(z)$ *is given by* $|x| + |y| \le |z|$ *for* $z \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$.

6) The space integral $\int_A yz\,d\Omega$, *where* $B(z)$ *is given by* $0 \le y \le x \le 1 - \frac{z}{2}$ *for* $z \in [0,2]$.
[Cf. **Example 1.1.6**.]

7) The space integral $\int_A xz\,d\Omega$, *where* $B(z)$ *is given by* $0 \le x \le 1$ *and* $0 \le y \le 1 - z$ *for* $z \in [0,1]$.
[Cf. **Example 1.1.7**.]

8) The space integral $\int_A z\,d\Omega$, *where* $B(z)$ *is given by* $x^2 + y^2 \le (2-z)^2$ *for* $z \in [0,2]$.
[Cf. **Example 1.1.8**.]

A Space integrals in rectangular coordinates, where the domain is sliced at height, $B(z)$.

D Whenever it is necessary, sketch $B(z)$. Then apply the second theorem of reduction.

I 1) Here,

$$B(z) = \{(x,y) \mid -z \le x \le z,\ -2z \le y \le 2z\} = [-z,z] \times [-2z,2z],$$

which is a rectangle for every $z \in]0,1]$ of the area

$$\text{area}\{B(z)\} = 8z^2.$$

We get by reduction,

$$\int_A z^2\,d\Omega = \int_0^1 x^2 \left\{ \int_{B(z)} dxdy \right\} dz = \int_0^1 z^2 \cdot \text{area}\{B(z)\}\,dz = \int_0^1 8z^4\,dz = \frac{8}{5}.$$

2) Here

$$B(z)=\{(x,t)\mid 0\le x,\,0\le y,\,x+y\le z^2\}$$

is a triangle for every $z\in]0,1]$, namely the lower triangle of the square $[0,z^2]\times[0,z^2]$, when this is cut by a diagonal from the upper left corner to the lower right corner.
We get by the theorem of reduction,

$$\begin{aligned}\int_A xz\,d\Omega &= \int_0^1 z\left\{\int_{B(z)} x\,dxdy\right\}dz=\int_0^1 z\left\{\int_0^{z^2} x\left[\int_0^{z^2-x} dy\right]dx\right\}dz\\ &= \int_0^1 z\left\{\int_0^{z^2}(xz^2-x^2)\,dx\right\}dz=\int_0^1 z\left[\frac{1}{2}\,x^2z^2-\frac{1}{3}\,x^3\right]_0^{z^2}dz=\int_0^1\frac{1}{6}\,z^7\,dz=\frac{1}{48}.\end{aligned}$$

3) Here

$$B(z)=\{(x,y)\mid 0\le x,\,0\le y,\,x+y\le 2-z\}$$

is a triangle for every $z\in[1.2[$, namely the lower triangle of the square $[0,2-z]\times[0,2-z]$, when this is cut by a diagonal from the upper left corner to the lower right corner.
Then by the theorem of reduction,

$$\begin{aligned}\int_A xy^2z\,d\Omega &= \int_1^2 z\left\{\int_{B(z)} xy^2\,dxdy\right\}dz=\int_1^2 z\left\{\int_0^{2-z} y^2\left[\int_0^{2-z-y} x\,dx\right]dy\right\}dz\\ &= \frac{1}{2}\int_1^2 z\left\{\int_0^{2-z} y^2(z-2+y)^2\,dy\right\}dz\\ &= \frac{1}{2}\int_1^2\{(z-2)+2\}\left\{\int_0^{2-z}\left[(z-2)^2y^2+2(z-2)y^3+y^4\right]dy\right\}dz\\ &= \frac{1}{2}\int_1^2\{(z-2)+2\}\left[\frac{1}{3}(z-2)^2y^3+\frac{1}{2}(z-2)y^4+\frac{1}{5}\,y^5\right]_{y=0}^{2-z}dz\\ &= \frac{1}{2}\int_1^2\{(z-2)+2)\}\cdot\left(-\frac{1}{3}+\frac{1}{2}-\frac{1}{5}\right)(z-2)^5\,dz\\ &= -\frac{1}{60}\int_1^2\left\{(z-2)^6+2(z-2)^5\right\}dz\\ &= -\frac{1}{60}\left[\frac{1}{7}\,(z-2)^7+\frac{1}{3}\,(z-2)^6\right]_1^2=-\frac{1}{60}\left(\frac{1}{7}-\frac{1}{3}\right)=\frac{4}{3\cdot 7\cdot 60}=\frac{1}{315}.\end{aligned}$$

4) Here

$$B(z)=\{(x,y)\mid 1\le x\le z,\,z\le y\le 2z\},\qquad z\in[1,3],$$

which is sketched on the figure.
We get by the theorem of reduction,

$$\begin{aligned}\int_A\frac{1}{xy^2}\,d\Omega &= \int_1^3\left\{\int_{B(z)}\frac{1}{xy^2}\,dxdy\right\}dz=\int_1^3\left\{\int_1^2\frac{1}{x}\left[\int_z^{2z}\frac{1}{y^2}\,dy\right]dx\right\}dz\\ &= \int_1^3\left\{\int_1^z\frac{1}{x}\left[-\frac{1}{y}\right]_z^{2z}dx\right\}dz=\int_1^3\frac{1}{2z}\,[\ln x]_{x=1}^z dz=\frac{1}{4}\left[(\ln z)^2\right]_1^3=\frac{1}{4}\,(\ln 3)^2.\end{aligned}$$

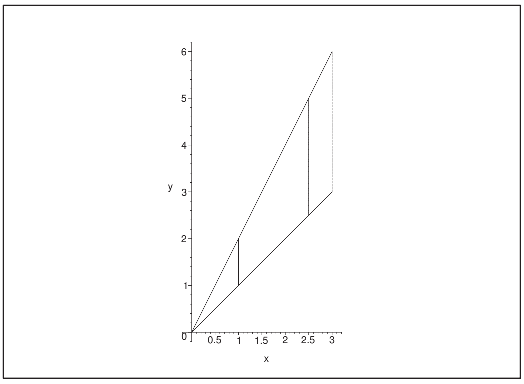

Figure 7: The domain $B(z)$ of **Example 1.2.4**.

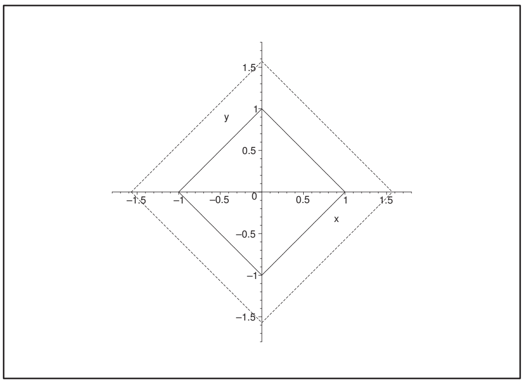

Figure 8: The domain $B(z)$ of **Example 1.2.5**.

5) By a continuous extension the integrand is put equal to 1 for $z = 0$. Note that $B(z)$ is a square of edge length $\sqrt{2}|z|$, hence of the area

$$\text{area}\{B(z)\} = 2z^2.$$

Then by the theorem of reduction,

$$\begin{aligned}
\int_A \left(\frac{\sin z}{z}\right)^2 d\Omega &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\sin z}{z}\right)^2 \left\{\int_{B(z)} dxdy\right\} dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\sin z}{z}\right) \text{areal}\{B(z)\}\, dz \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\sin z}{z}\right)^2 \cdot 2z^2\, dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\sin^2 z\, dz \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\cos 2z)\, dz = \pi - \left[\frac{1}{2}\sin 2z\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi.
\end{aligned}$$

6) Here

$$B(z) = \left\{(x,y) \;\middle|\; 0 \le y \le x \le 1 - \frac{z}{2}\right\}$$

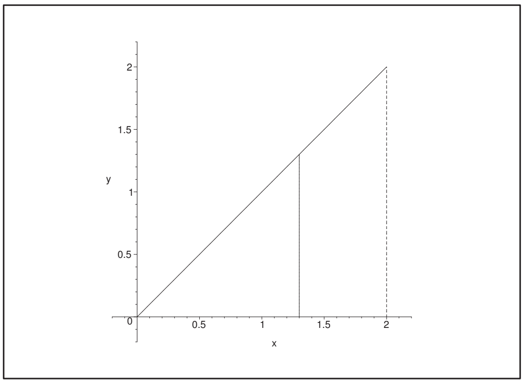

Figure 9: The domain $B(z)$ of **Example 1.2.6**.

is a triangle for every $z \in [0,2[$.

We get by the second theorem of reduction,

$$\begin{aligned}
\int_A yz\,d\Omega &= \int_0^2 z\left\{\int_{B(z)} y\,dS\right\}dz = \int_0^2 z\left\{\int_0^{1-\frac{z}{2}}\left(\int_0^x y\,dy\right)dx\right\}dz \\
&= \int_0^2 z\left\{\int_0^{1-\frac{z}{2}} \frac{1}{2}\,x^2\,dx\right\}dz = \frac{1}{6}\int_0^2 z\left(-\frac{z}{2}\right)^3 dz \\
&= \frac{1}{6}\int_0^2 z\left(1-\frac{3}{2}\,z+\frac{3}{4}\,z^2-\frac{1}{8}\,z^3\right)dz = \frac{1}{6}\int_0^2\left(z-\frac{3}{2}\,z^2+\frac{3}{4}\,z^3-\frac{1}{8}\,z^4\right)dz \\
&= \frac{1}{6}\left[\frac{1}{2}\,z^2-\frac{1}{2}\,z^3+\frac{3}{16}\,z^4-\frac{1}{40}\,z^5\right]_0^2 = \frac{1}{6}\left(\frac{4}{2}-\frac{8}{4}+\frac{3}{16}\cdot 16-\frac{1}{40}\cdot 32\right) \\
&= \frac{1}{6}\left(2-4+3-\frac{4}{5}\right) = \frac{1}{6}\left(1-\frac{4}{5}\right) = \frac{1}{30}.
\end{aligned}$$

REMARK. The domain is also described by

$$0 \le x \le 1, \quad 0 \le y \le x, \quad 0 \le z \le 2-2x,$$

cf. **Example 1.1.6**, and we have computed the integral in two different ways (and luckily obtained the same result). $\Diamond$

7) The have the same integrand and the same domain as in **Example 1.1.7**, so we must get the same result. The only difference is that we here cut the domain into slices, while we in i **Example 1.1** used the "method of upright posts".

We get by the theorem of reduction,

$$\begin{aligned}
\int_A xz\,d\Omega &= \int_0^1 x\,dx\cdot\int_0^1 z\left\{\int_0^{1-z} dy\right\}dz = \frac{1}{2}\int_0^1 z(1-z)\,dz \\
&= \frac{1}{2}\int_0^1\{z-z^2\}dz = \frac{1}{2}\left\{\frac{1}{2}-\frac{1}{3}\right\} = \frac{1}{2}\cdot\frac{1}{6} = \frac{1}{12}.
\end{aligned}$$

8) We have the same integrand and the same set as in **Example 1.1.8**, so we must get the same result. The only difference is that we here cut the domain into slices, while we in **Example 1.1** used the "method of upright posts". Also note that we use polar coordinates in each slice, so the example should actually be moved to **Example 3.1**.

We get by the theorem of reduction in semi-polar coordinates and the change of variables $u = 2 - z$ that

$$\begin{aligned}\int_A z\,d\Omega &= \int_0^2 z\cdot\pi(2-z^2)\,dz = \pi\int_0^2 (2-u)u^2\,du = \pi\int_0^2 (2u^2-u^3)\,du = \pi\left[\frac{2}{3}\,u^3 - \frac{1}{4}\,u^4\right] \\ &= \pi\left\{\frac{16}{3} - \frac{16}{4}\right\} = \frac{16}{12}\,\pi = \frac{4\pi}{3}.\end{aligned}$$

Example 1.3 *Let A be the tetrahedron of the vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Compute in each of the following cases the space integral*

$$\int_A f(x,y,z)\,d\Omega,$$

where

1) $f(x,y,z) = x+y+z$,

2) $f(x,y,z) = \cos(x+y+z)$,

3) $f(x,y,z) = \exp(x+y+z)$,

4) $f(x,y,z) = (1+x+y+z)^{-3}$,

5) $f(x,y,z) = x^2+y^2+z^2$,

6) $f(x,y,z) = xy - yz$.

A Space integrals over a tetrahedron.

D Consider the tetrahedron as a cone with $(0,0,0)$ as its top point in the first four questions, where the natural variable is $x+y+z$. Therefore, first analyze this special case. Compute the space integral with respect to this variable. Alternatively, compute the triple integral. There is also the possibility of some arguments of symmetry.

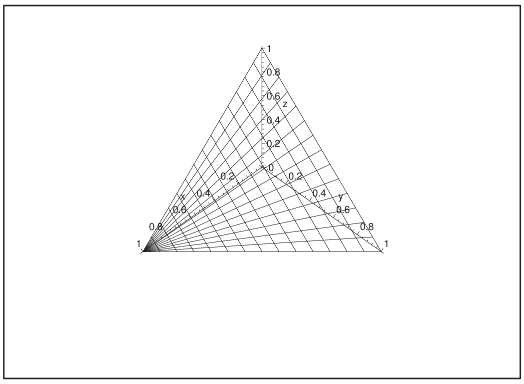

Figure 10: The tetrahedron of the vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

I Preparations. The distance from $(0,0,0)$ to the plane $x+y+z=1$ is

$$\sqrt{\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2} = \frac{1}{\sqrt{3}}.$$

Thus we can consider the tetrahedron as a cone of height $h = \dfrac{1}{\sqrt{3}}$ and with the surface where $x+y+z=1$ as its base. the area of this base is

$$\frac{1}{2}\,|(1,0,0)-(0,1,0)|\cdot\left|(0,0,1)-\left(\frac{1}{2},\frac{1}{2},0\right)\right| = \frac{1}{2}\,\sqrt{2}\cdot\sqrt{\frac{1}{4}+\frac{1}{4}+1} = \frac{1}{2}\,\sqrt{2}\cdot\sqrt{\frac{3}{2}} = \frac{\sqrt{3}}{2}.$$

Intersect the tetrahedron by the plane $x+y+z=t$, $t \in [0,1]$, parallel to the base. Then the distance from the new triangle $B(t)$ to the top point $(0,0,0)$ is $\dfrac{t}{\sqrt{3}}$, thus the area of this triangle $B(t)$ is due to the similarity given by

$$\operatorname{area}(B(t)) = \left(\frac{1/\sqrt{3}}{1/\sqrt{3}}\right)^2 t^2\,\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}\,t^2, \qquad t \in [0,1].$$

If the integrand $f(x,y,z) = g(x+y+z)$ is a function in $t = x+y+z$, we even get the simpler formula

$$(1)\quad \int_A f(x,y,z)\,d\Omega = \frac{1}{\sqrt{3}}\int_0^1 g(t)\operatorname{area}(B(t))\,dt = \frac{1}{2}\int_0^1 t^2 g(t)\,dt.$$

Clearly, (1) can be applied in the first four questions.

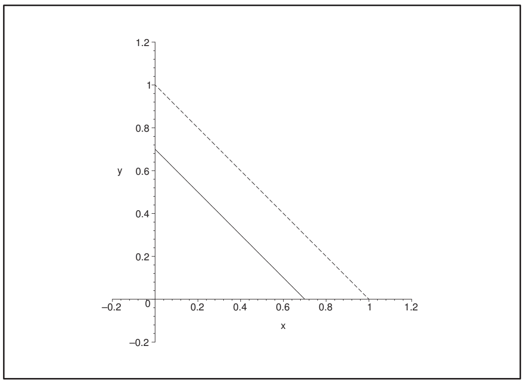

Figure 11: The projection of $B'(z)$ onto the XY-plane.

1) From $f(x,y,z) = x+y+z = t = g(t)$ and (1) follows that

$$\int_A (x+y+z)\,d\Omega = \frac{1}{2}\int_0^1 t^2 g(t)\,dt = \frac{1}{2}\int_0^1 t^3\,dt = \frac{1}{8}.$$

ALTERNATIVELY the plane $z = \text{constant}$, $z \in [0,1[$, intersects the tetrahedron in a set, the projection of which onto the XY-plane is

$$B'(z) = \{(x,y) \mid x \geq 0,\ y \geq 0,\ x+y \leq 1-z\}.$$

Hence by more traditional calculations,

$$\int_A (x+y+z)\,d\Omega = \int_0^1 z \cdot \operatorname{area}(B'(z))\,dz + \int_0^1 \left\{\int_{B'(z)} (x+y)\,dx\,dy\right\} dz.$$

It follows by the symmetry that

$$\int_{B'(z)} x\,dx\,dy = \int_{B'(z)} y\,dx\,dy,$$

hence

$$\begin{aligned}
\int_A (x+y+z)\,d\Omega &= \int_0^1 z\cdot\frac{1}{2}(1-z)^2dz+2\int_0^1\left\{\int_{B'(z)} x\,dx\,dy\right\}dz\\
&= \frac{1}{2}\int_0^1(1-t)t^2dt+2\int_0^1\left\{\int_0^{1-z} x\left\{\int_0^{1-x-z} dy\right\}dx\right\}dz\\
&= \frac{1}{2}\left[\frac{1}{3}t^3-\frac{1}{4}t^4\right]_0^1+2\int_0^1\left\{\int_0^{1-z} x[(1-z)-x]dx\right\}dz\\
&= \frac{1}{24}+2\int_0^1\left[\frac{1}{2}x^2(1-z)-\frac{1}{3}x^3\right]_0^{1-z}dz\\
&= \frac{1}{24}+\frac{2}{6}\int_0^1(1-z)^3dz=\frac{1}{24}+\frac{1}{12}\left]-(1-z)^4\right]_0^1=\frac{1}{24}+\frac{1}{12}=\frac{1}{8}.
\end{aligned}$$

2) It follows from $f(x,y,z)=\cos(x+y+z)=\cos t=g(t)$ and (1) that

$$\begin{aligned}
\int_A \cos(x+y+z)\,d\Omega &= \frac{1}{2}\int_0^1 t^2\cos t\,dt=\frac{1}{2}\left[t^2\sin t+2t\cos t-2\sin t\right]_0^1\\
&= \frac{1}{2}\sin 1+\cos 1-\sin 1=\cos 1-\frac{1}{2}\sin 1.
\end{aligned}$$

ALTERNATIVELY, we get by more traditional calculations, where we use the same set $B'(z)$ as in 1),

$$\begin{aligned}
\int_A \cos(x+y+z)\,d\Omega &= \int_0^1 \left\{ \int_{B'(z)} \cos(x+y+z)\,dx\,dy \right\} dz \\
&= \int_0^1 \left\{ \int_0^{1-z} \left\{ \int_0^{1-z-x} \cos(x+y+z)\,dy \right\} dx \right\} dz \\
&= \int_0^1 \left\{ \int_0^{1-z} [\sin(x+y+z)]_{y=0}^{1-z-x} dx \right\} dz = \int_0^1 \left\{ \int_0^{1-z} \{\sin 1 - \sin(x+z)\} dx \right\} dz \\
&= \sin 1 \cdot \int_0^1 (1-z)\,dz + \int_0^1 [\cos(x+z)]_{x=0}^{1-z} dz = \frac{1}{2}\,\sin 1 + \int_0^1 \{\cos 1 - \cos z\} dz \\
&= \frac{1}{2}\,\sin 1 + \cos 1 - \sin 1 = \cos 1 - \frac{1}{2}\,\sin 1.
\end{aligned}$$

3) It follows from $f(x,y,z) = \exp(x+y+z) = e^t = g(t)$ and (1) that

$$\begin{aligned}
\int_A \exp(x+y+z)\,d\Omega &= \frac{1}{2}\int_0^1 t^2 e^t\,dt = \frac{1}{2}\left[t^2e^t - 2e^t + 2e^t\right]_0^1 \\
&= \frac{1}{2}\,(e - 2e + 2e - 2) = \frac{1}{2}\,(e-2).
\end{aligned}$$

ALTERNATIVELY, by traditional computations,

$$\begin{aligned}
\int_A \exp(x+y+z)\,d\Omega &= \int_0^1 \left\{ \int_0^{1-z} \left\{ \int_0^{1-z-x} \exp(x+y+z)\,dy \right\} dx \right\} dz \\
&= \int_0^1 \left\{ \int_0^{1-z} [\exp(x+y+z)]_{y=0}^{1-z-x} dx \right\} dz = \int_0^1 \left\{ \int_0^{1-z} \left(e - e^{x+z}\right)\,dx \right\} dz \\
&= e\int_0^1 \left\{ \int_0^{1-z} dx \right\} dz - \int_0^1 \left\{ \int_0^{1-z} e^{x+z}\,dx \right\} dz \\
&= e\int_0^1 (1-z)\,dz - \int_0^1 \left[e^{x+z}\right]_{x=0} 1 - z dz = \frac{1}{2}\,e - \int_0^1 (e - e^z)\,dz \\
&= \frac{1}{2}\,e - e + [e^z]_0^1 = \frac{e}{2} - 1 = \frac{1}{2}\,(e-2).
\end{aligned}$$

4) From $f(x,y,z) = (1+x+y+z)^{-3} = (1+t)^{-3} = g(t)$ and (1) follows that

$$\begin{aligned}
\int_A (1+x+y+z)^{-3}\,\mathrm{d}\Omega &= \frac{1}{2}\int_0^1 \frac{t^2}{(1+t)^3}\,dt = \frac{1}{2}\int_0^1 \left\{ \frac{1}{(t+1)^3} - \frac{2}{(t+1)^2} + \frac{1}{t+1} \right\} dt \\
&= \frac{1}{2}\left[-\frac{1}{2}\cdot\frac{1}{(t+1)^2} + \frac{2}{t+1} + \ln(t+1) \right]_0^1 = \frac{1}{2}\left\{ -\frac{1}{2}\cdot\frac{1}{4} + \frac{2}{2} + \ln 2 + \frac{1}{2} - 2 \right\} \\
&= \frac{1}{2}\left\{ \ln 2 - \frac{5}{8} \right\} = \frac{1}{2}\,\ln 2 - \frac{5}{16}.
\end{aligned}$$

ALTERNATIVELY, by traditional calculations,

$$\int_A (1+x+y+z)^{-3}\,\mathrm{d}\Omega = \int_0^1 \left\{ \int_0^{1-z} \left\{ \int_0^{1-z-x} (1+x+y+z)^{-3}\,dy \right\} dx \right\} dz$$

$$= \int_0^1 \left\{ \int_0^{1-z} \left[-\frac{1}{2}\,(1+x+y+z)^{-2} \right]_{y=0}^{1-z-x} dx \right\} dz$$

$$= \frac{1}{2} \int_0^1 \left\{ \int_0^{1-z} \left\{ (1+x+z)^{-2} - \frac{1}{4} \right\} dx \right\} dz$$

$$= \frac{1}{2} \int_0^1 \left[-(1+x+z)^{-1} \right]_{x=0}^{1-z} dz - \frac{1}{8} \int_0^1 \left\{ \int_0^{1-z} dx \right\} dz$$

$$= \frac{1}{2} \int_0^1 \left\{ \frac{1}{1+z} - \frac{1}{2} \right\} dz - \frac{1}{16} = \frac{1}{2}\,\ln 2 - \frac{1}{4} - \frac{1}{16} = \frac{1}{2}\,\ln 2 - \frac{5}{16}.$$

5) In this case we can no longer apply (1). We note by symmetry that

$$\int_{B'(z)} x^2\,dx\,dy = \int_{B'(z)} y^2\,dx\,dy,$$

hence by traditional calculations,

$$\int_A (x^2+y^2+z^2)\,\mathrm{d}\Omega = \int_0^1 z^2\,\mathrm{areal}(B'(z))\,dz + 2 \int_0^1 \left\{ \int_{B'(z)} x^2\,dx\,dy \right\} dz$$

$$= \frac{1}{2} \int_0^1 z^2(1-z)^2\,dz + 2 \int_0^1 \left\{ \int_0^{1-z} x^2 \left\{ \int_0^{1-z-x} dy \right\} dx \right\} dz$$

$$= \frac{1}{2} \int_0^1 (z^2 - 2z^3 + z^4)\,dz + 2 \int_0^1 \left\{ \int_0^{1-z} x^2(1-z-x)\,dx \right\} dz$$

$$= \frac{1}{2} \left[\frac{1}{3}\,z^3 1 - \frac{2}{4}\,z^4 + \frac{1}{5}\,z^5 \right]_0^1 + 2 \int_0^1 \left[\frac{1}{3}\,x^3(1-z) - \frac{1}{4}\,x^4 \right]_{x=0}^{1-z} dz$$

$$= \frac{1}{2} \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right) + 2 \cdot \frac{1}{12} \int_0^1 (1-z)^4\,dz$$

$$= \frac{1}{60}\,(10-15+6) + \frac{1}{6} \left[-\frac{1}{5}(1-z)^5 \right]_0^1 = \frac{1}{60} + \frac{1}{30} = \frac{1}{20}.$$

6) Put

$$B''(y) = \{(x,z) \mid 0 \le x,\ 0 \le z,\ x+z \le 1-y\}.$$

It follows fore symmetric reasons that

$$\int_{B''(y)} xy\,dx\,dz = \int_{B''(y)} yz\,dx\,dz.$$

Hence

$$\int_A (xy - yz)\,d\Omega = \int_0^1 \left\{ \int_{B''(y)} (xy - yz)\,dx\,dz \right\} dy = 0.$$

ALTERNATIVELY we get by traditional calculations,

$$\int_A (xy-yz)\,d\Omega = \int_0^1 \left\{ \int_{B'(z)} (x-z)y\,dx\,dy \right\} dz$$
$$= \int_0^1 \left\{ \int_0^{1-z} (x-z) \left\{ \int_0^{1-(x+z)} y\,dy \right\} dx \right\} dz = \frac{1}{2}\int_0^1 \left\{ \int_0^{1-z} (x-z)[1-(x+z)]^2 dx \right\} dz$$
$$= \frac{1}{2}\int_0^1 \left\{ \int_0^{1-z} (x+z)[(x+z)-1]^2 dx \right\} dz - \int_0^1 z \left\{ \int_0^{1-z} [(x+z)-1]^2\,dx \right\} dz.$$

In order to avoid too complicated expressions we compute the two double integrals one by one:

$$\frac{1}{2}\int_0^1 \left\{ \int_0^{1-z} (x+z)[(x+z)-1]^2 dx \right\} dz = \frac{1}{2}\int_0^1 \left\{ \int_0^{1-z} \{(x+z)^3 - 2(x+z)^2 + (x+z)\} dx \right\} dz$$
$$= \frac{1}{2}\int_0^1 \left[\frac{1}{4}(x+z)^4 - \frac{2}{3}(x+z)^3 + \frac{1}{2}(x+z)^2 \right]_{x=0}^{1-z} dz = \frac{1}{2}\int_0^1 \left\{ \frac{1}{4} - \frac{2}{3} + \frac{1}{2} - \frac{1}{4}z^4 + \frac{2}{3}z^3 - \frac{1}{2}z^2 \right\} dz$$
$$= \frac{1}{24} + \frac{1}{2}\left[-\frac{1}{20} + \frac{1}{6} - \frac{1}{6} \right] = \frac{1}{24} - \frac{1}{40} = \frac{5-3}{120} = \frac{1}{60},$$

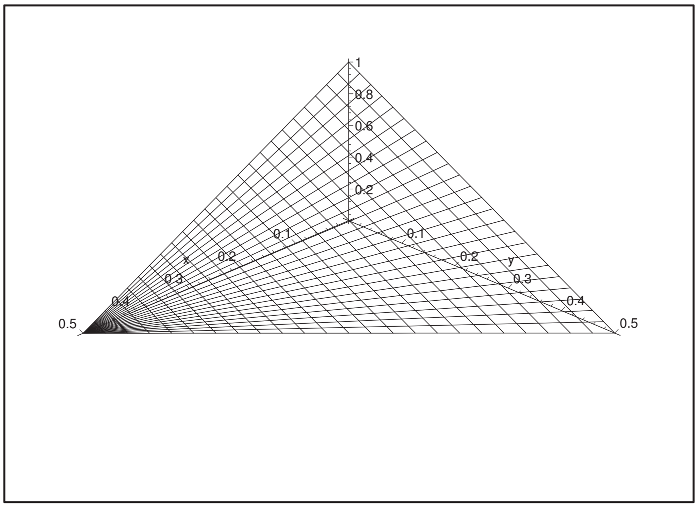

Figure 12: The tetrahedron A with its projection B onto the (x, y)-plane.

and

$$\int_0^1 z\left\{\int_0^{1-z}[(x+z)-1]^2\,dx\right\}dz = \int_0^1 z\left[\frac{1}{3}(x+z-1)^3\right]_{x=0}^{1-z} dz$$
$$= \frac{1}{3}\int_0^1 z(1-z)^3\,dz = \frac{1}{3}\int_0^1 (1-t)t^3\,dt = \frac{1}{3}\left[\frac{1}{4}\,t^4 - \frac{1}{5}\,t^5\right]_0^1 = \frac{1}{60}.$$

Finally, we get by insertion,

$$\int_A (xy - yz)\,d\Omega = \frac{1}{60} - \frac{1}{60} = 0.$$

Example 1.4 *Let B be the triangle which is bounded by the X-axis and the Y-axis and the line of the equation $x + y = \frac{1}{2}$. Furthermore, let A be the tetrahedron bounded by the three coordinate planes and the plane of the equation $2x + 2y + z = 1$. Compute the integrals*

$$\int_B (1 - 2x - 2y)\,dx\,dy \quad \textit{and} \quad \int_A (x + y + z)\,dx\,dy\,dz.$$

A Plane integral and space integral.

D Sketch B and A. Then compute the integrals.

I It follows immediately that B is that surfaces of A, which lies in the i (x, y)-plane.

First calculate the plane integral (it is actually the volume of the tetrahedron A),

$$\begin{aligned}
\int_B (1-2x-2y)dx\,dy &= \int_0^{\frac{1}{2}} \left\{ \int_0^{\frac{1}{2}-x} (1-2x-2y)\,dy \right\} dx \\
&= \int_0^{\frac{1}{2}} \left\{ (1-2x)\left(\frac{1}{2}-x\right) - \left(\frac{1}{2}-x\right)^2 \right\} dx = \int_0^{\frac{1}{2}} \left(\frac{1}{2}-x\right)^2 dx \\
&= \left[\frac{1}{3}\left(x-\frac{1}{2}\right)^3 \right]_0^{\frac{1}{2}} = 0 - \frac{1}{3}\left(-\frac{1}{2}\right)^3 = \frac{1}{24}.
\end{aligned}$$

Then compute the space integral,

$$\begin{aligned}
\int_A (x+y+z)\,dx\,dy\,dz &= \int_B \left\{ \int_0^{1-2x-2y} (x+y+z)\,dz \right\} dx\,dy \\
&= \int_B \left\{ (x+y)(1-2x-2y) + \frac{1}{2}(1-2x-2y)^2 \right\} dx\,dy \\
&= \frac{1}{2} \int_B (1-2x-2y)\{2x+2y+(1-2x-2y)\}\,dx\,dy \\
&= \frac{1}{2} \int_B (1-2x-2y)\,dx\,dy = \frac{1}{2}\cdot\frac{1}{24} = \frac{1}{48},
\end{aligned}$$

where we have inserted the value of the plane integral.

Example 1.5 *Consider two balls and their intersection*

$$\Omega_1 = \overline{K}((0,0,0);a), \qquad \Omega_2 = \overline{K}\left((0,0,a);\frac{a}{2}\right), \qquad \Omega = \Omega_1 \cap \Omega_2.$$

1) *Sketch the three point sets by means of a meridian half plane, and describe the position of the intersection circle* $\partial\Omega_1 \cap \partial\Omega_2$.

2) *Find the volume of* Ω.

3) *Compute the space integral*

$$\int_\Omega (2-xy)\,d\Omega.$$

A Space integrals.

D Follow the given guidelines.

I 1) The two circles cut each other at height $z \in \left]\frac{a}{2}, a\right[$. Then by Pythagoras's theorem,

$$r^2 = a^2 - z^2 = \left(\frac{a}{2}\right)^2 - (a-z)^2 = -\frac{3}{4}\,a^2 + 2az - z^2.$$

A reduction gives $2az = \dfrac{7}{4}\,a^2$, thus $z = \dfrac{7}{8}\,a$, which indicates the whereabouts of the plane, in which the intersection circle $\partial\Omega_1 \cap \partial\Omega_2$ lies.

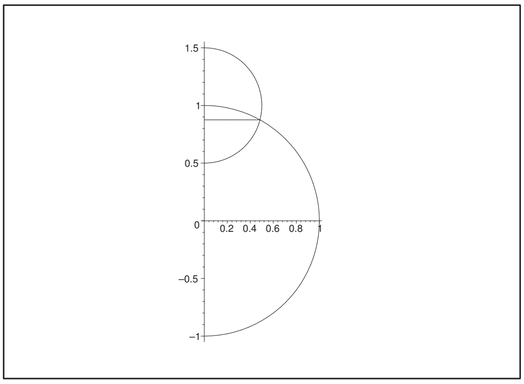

Figure 13: The situation in the meridian half plane for $a = 1$.

2) Then split $\Omega = \omega_1 \cup \omega_2$ into its two natural subregions, where ω_1 lies above the plane $z = \dfrac{7}{8}\, a$, and ω_2 lies below the same plane. We use in each of the subregions ω_1 and ω_2 the "method of slices", where each slice is parallel to the (x, y)-plane. By translating the subregion ω_2 in a convenient way we finally get

$$
\begin{aligned}
\mathrm{vol}(\Omega) &= \mathrm{vol}(\omega_1) + \mathrm{vol}(\omega_2) = \int_{\frac{7}{8}\,a}^{a} \pi\left(a^2 - z^2\right)\, dz + \int_{-\frac{1}{2}\,a}^{-\frac{1}{8}\,a} \pi \left(\frac{a^2}{4} - z^2\right) dz \\
&= \pi\left[a^2 z - \frac{1}{3}\,z^3\right]_{\frac{7}{8}\,a}^{a} + \pi\left[\frac{a^2}{4}\,z - \frac{1}{3}\,z^3\right]_{\frac{1}{8}\,a}^{\frac{1}{2}\,a} \\
&= \pi a^3 \left\{\left(1 - \frac{1}{3} - \frac{7}{8} + \frac{1}{3}\cdot\left(\frac{7}{8}\right)^3\right) + \left(\frac{1}{4}\cdot\frac{1}{2} - \frac{1}{3}\cdot\frac{1}{8} - \frac{1}{4}\cdot\frac{1}{8} + \frac{1}{3}\cdot\left(\frac{1}{8}\right)^3\right)\right\} \\
&= \frac{\pi a^3}{8}\left\{1 - \frac{8}{3} + \frac{7}{3}\cdot\left(\frac{7}{8}\right)^2 + 1 - \frac{1}{3} - \frac{1}{4} + \frac{1}{3}\cdot\frac{1}{8^2}\right\} \\
&= \frac{\pi a^3}{24}\left\{-3 - \frac{3}{4} + \frac{1}{64}\,(343 + 1)\right\} = \frac{\pi a^3}{24}\left\{-\frac{15}{4} + \frac{43}{8}\right\} = \frac{13}{192}\,\pi a^3.
\end{aligned}
$$

3) Of symmetric reasons, $\int_\Omega xy\, d\Omega = 0$, thus

$$\int_\Omega (2 - xy)\, d\Omega = 2 \cdot \mathrm{vol}(\Omega) = \frac{13}{96}\,\pi a^3.$$

Example 1.6 *Given the tetrahedron*

$$T = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \le x,\ 0 \le y.\ 0 \le z,\ z + 2x + 4y \le 8\}.$$

Compute the space integral

$$\int_T x\, d\Omega.$$

A Space integral.

D First find the base of T in the (x, y)-plane.

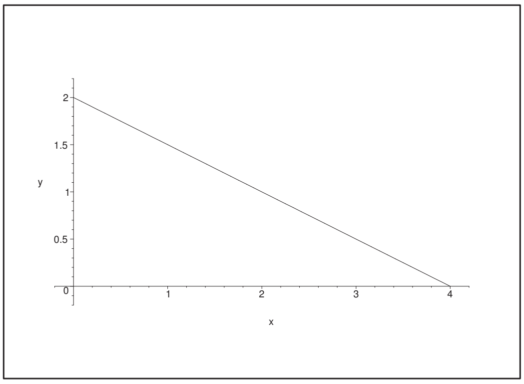

Figure 14: The base B of T in the plane $z = 0$.

I The base B is given by

$$0 \le x, \qquad 0 \le y, \qquad 2x + 4y \le 8,$$

i.e.

$$B = \{(x,y) \mid 0 \le x,\, 0 \le y,\, x + 2y \le 4\}.$$

Then we get the space integral

$$\begin{aligned}
\int_T x\, d\Omega &= \int_0^4 \left\{ \int_0^{\frac{1}{2}(4-x)} x \cdot (8 - 2x - 4y)\, dy \right\} dx \\
&= -\frac{1}{8} \int_0^4 x \left[(8 - 2x - 4y)^2 \right]_{y=0}^{\frac{1}{2}(4-x)} dx = \frac{1}{8} \int_0^4 x(8 - 2x)^2\, dx \\
&= \frac{4}{8} \int_0^4 \{(x-4) + 4\}(x-4)^2\, dx = \frac{1}{2} \left[\frac{1}{4}(x-4)^4 + \frac{4}{3}(x-4)^3 \right]_0^4 \\
&= \frac{1}{2} \left\{ -\frac{1}{4} \cdot 4^4 + \frac{4}{3} \cdot 4^3 \right\} = \frac{4^3}{2} \left(\frac{4}{3} - 1 \right) = \frac{32}{3}.
\end{aligned}$$

ALTERNATIVELY, start by integrating with respect to x. Then

$$\begin{aligned}
\int_T x\, d\Omega &= \int_0^2 \left\{ \int_0^{4-2y} (8x - 2x^2 - 4xy)\, dx \right\} dy = \int_0^2 \left[4x^2 - \frac{2}{3} s^3 - 2x^2 y \right]_{x=0}^{4-2y} dy \\
&= \int_0^2 \left\{ (4-2y) \cdot (4-2y)^2 - \frac{2}{3}(4-2y)^3 \right\} dy = \frac{1}{3} \int_0^2 (4-2y)^3\, dy = \frac{8}{3} \int_0^2 (2-y)^3\, dy \\
&= \frac{8}{3} \int_0^2 t^3\, dt = \frac{8}{3} \left[\frac{t^4}{4} \right]_0^2 = \frac{32}{3}.
\end{aligned}$$

Example 1.7 *Given a curve $\mathcal{K}$ in the (z, x)-plane of the equation*

$$x = \cos z, \qquad z \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

The curve $\mathcal{K}$ is rotated once around the z-axis in the (x, y, z)-space, creating the surface of revolution $\mathcal{F}$. Let A denote the bounded domain in the (x, y, z)-space with $\mathcal{F}$ as its boundary surface.

1) Find the volume of A.

2) Compute the space integral

$$\int_A \sqrt{x^2+y^2}\, d\Omega.$$

A Body of revolution an space integral.

D Sketch a figure and then just compute.

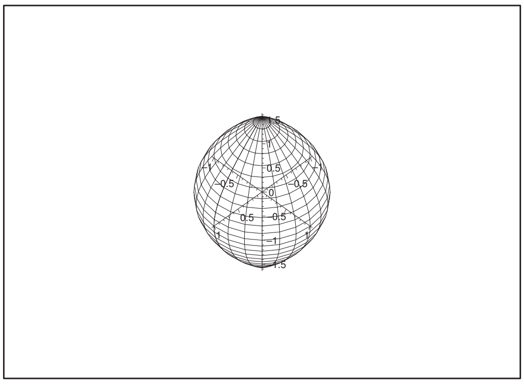

Figure 15: The domain A with the boundary surface $\mathcal{F}$.

I 1) The domain A is the spindle shaped body on the figure.

We get by slicing the body,

$$\mathrm{vol}(A) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi \cos^2 z\, dz = 2\pi \int_0^{\frac{\pi}{2}} \frac{1+\cos 2z}{2}\, dz = 2\pi \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{2}.$$

2) If we put

$$B_z = \{(x,y) \mid \sqrt{x^2+y^2} \le \cos z\}, \qquad z \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

then

$$\begin{aligned}
\int_A \sqrt{x^2+y^2}\, d\Omega &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ \int_{B_z} \sqrt{x^2+y^2}\, dx\, dy \right\} dz = 2\int_0^{\frac{\pi}{2}} \left\{ 2\pi \int_0^{\cos z} \varrho \cdot \varrho\, d\varrho \right\} dz \\
&= 4\pi \int_0^{\frac{\pi}{2}} \left[\frac{\varrho^3}{3}\right]_0^{\cos z} dz = \frac{4\pi}{3} \int_0^{\frac{\pi}{2}} \cos^3 z\, dz \\
&= \frac{4\pi}{3} \int_0^{\frac{\pi}{2}} (1-\sin^2 z)\cos z\, dz = \frac{4\pi}{3}\left[\sin z - \frac{1}{3}\sin^3 z\right]_0^{\frac{\pi}{2}} = \frac{8\pi}{9}.
\end{aligned}$$

Example 1.8 *Let a be a positive constant, and let*

$$A = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in B,\, \sqrt{ax} \le z \le \sqrt{ax+y^2}\},$$

where

$$B = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le a,\, -x \le y \le 2x\}.$$

Compute the space integral

$$\int_A xyz\, d\Omega.$$

A Space integral.

D Reduce the integral by first integrating with respect to z.

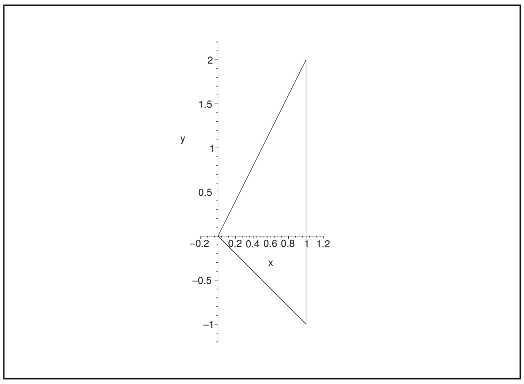

Figure 16: The domain B for $a = 1$.

I When we reduce as a triple integral, we get

$$\begin{aligned}
\int_A xyz\, d\Omega &= \int_0^a \left\{ \int_{-x}^{2x} \left(\int_{\sqrt{ax}}^{\sqrt{ax+y^2}} xyz\, dz \right) dy \right\} dx = \int_0^a x \left\{ \int_{-x}^{2x} y \left[\frac{1}{2} z^2 \right]_{\sqrt{ax}}^{\sqrt{ax+y^2}} dy \right\} dx \\
&= \frac{1}{2} \int_0^a x \left\{ \int_{-x}^{2x} y^3\, dy \right\} dx = \frac{1}{2} \int_0^a x \left[\frac{1}{4} y^4 \right]_{-x}^{2x} dx = \frac{1}{8} \int_0^a x \left\{ 2^4 - 1 \right\} x^4\, dx \\
&= \frac{15}{8} \int_0^a x^5\, dx = \frac{15}{8} \cdot \frac{a^6}{6} = \frac{5}{16}\, a^6.
\end{aligned}$$

Example 1.9 *Let c be a positive constant. Consider the half ball A given by the inequalities*

$$x^2 + y^2 + z^2 \leq c^2, \qquad z \geq 0.$$

1) Compute the space integral

$$J = \int_A z \, d\Omega.$$

2) Show that both the space integrals $\int_A x \, d\Omega$ and $\int_A y \, d\Omega$ are zero.

A Space integrals.

D Apply the slicing method and convenient symmetric arguments.

Alternatively, reduce in

1) spherical coordinates,
2) semi-polar coordinates,
3) rectangular coordinates.

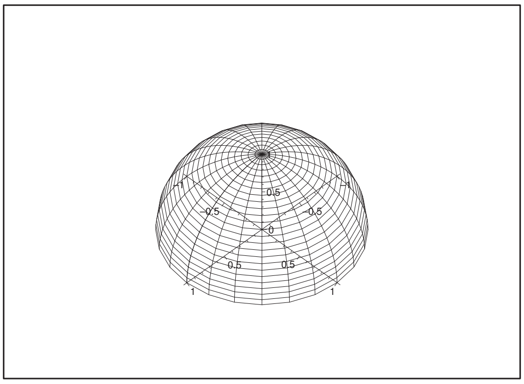

Figure 17: The half ball A for $c = 1$.

I 1) **First variant.** *The slicing method.*

At the height z the body A is cut into a disc $B(z)$ of radius $\sqrt{c^2 - z^2}$, hence of area $(c^2 - z^2)\pi$. Then we get by the slicing method,

$$J = \int_A z\, d\Omega = \int_0^c z \text{ area}(B(z))\, dz = \pi \int_0^c \{c^2 z - z^3\}\, dz = \pi \left[c^2 \cdot \frac{z^2}{2} - \frac{z^4}{4} \right]_0^c = \frac{\pi}{4}\, c^4.$$

Second variant. *Spherical coordinates.*

The set A is in spherical coordinates described by

$$\left\{ \begin{array}{l} x = r \sin\theta \cos\varphi, \\ \\ y = r \sin\theta \sin\varphi, \\ \\ z = r \cos\theta, \end{array} \right. \qquad \left\{ \begin{array}{l} \varphi \in [0, 2\pi], \\ \\ \theta \in \left[0, \dfrac{\pi}{2}\right], \\ \\ r \in [0, c], \end{array} \right.$$

and $d\Omega = r^2 \sin\theta\, dr\, d\theta\, d\varphi$. Thus we get by reduction

$$\begin{aligned} J &= \int_A z\, d\Omega = \int_0^{2\pi} \left\{ \int_0^{\frac{\pi}{2}} \left(\int_0^c r \cos\theta \cdot r^2 \sin\theta\, dr \right) d\theta \right\} d\varphi \\ &= 2\pi \cdot \left[\frac{1}{2} \sin^2\theta \right]_0^{\frac{\pi}{2}} \cdot \left[\frac{r^4}{4} \right]_0^c = 2\pi \cdot \frac{1}{2} \cdot \frac{1}{4}\, c^4 = \frac{\pi}{4}\, c^4. \end{aligned}$$

Third variant. *Semi-polar coordinates.*

In semi-polar coordinates A is described by

$$\left\{ \begin{array}{l} x = \varrho \cos\varphi, \\ \\ y = \varrho \sin\varphi, \\ \\ z = z, \end{array} \right. \qquad \left\{ \begin{array}{l} \varphi \in [0, 2\pi], \\ \\ z \in [0, c], \\ \\ \varrho \in \left[0, \sqrt{c^2 - z^2}\right], \end{array} \right.$$

and $d\Omega = \varrho\, d\varrho\, d\varphi\, dz$. We therefore get by reduction

$$\begin{aligned} J &= \int_A z\, d\Omega = \int_0^{2\pi} \left\{ \int_0^c \left(\int_0^{\sqrt{c^2-z^2}} z \cdot \varrho\, d\varrho \right) dz \right\} d\varphi \\ &= 2\pi \int_0^c z \left[\frac{1}{2}\,\varrho^2 \right]_0^{\sqrt{c^2-z^2}} dz = \pi \int_0^c \left(c^2 z - z^3\right)\, dz = \frac{\pi}{4}\, c^4. \end{aligned}$$

Fourth variant. *Rectangular coordinates.*
Here A is described by

$$0 \le z \le c, \quad |x| \le \sqrt{c^2-z^2}, \quad |y| \le \sqrt{c^2-z^2-x^2},$$

hence

$$\begin{aligned} J &= \int_A z\, d\Omega = \int_0^c z \left\{ \int_{-\sqrt{c^2-z^2}}^{\sqrt{c^2-z^2}} \left(\int_{-\sqrt{c^2-z^2-x^2}}^{\sqrt{c^2-z^2-x^2}} dy \right) dx \right\} dz \\ &= 2 \int_0^c z \left\{ \int_{-\sqrt{c^2-x^2}}^{\sqrt{c^2-z^2}} \sqrt{c^2-z^2-x^2}\, dx \right\} dz. \end{aligned}$$

We then get by the substitution $x = \sqrt{c^2-z^2} \cdot t$,

$$\begin{aligned} J &= 4 \int_0^c z \left\{ \int_0^{\sqrt{c^2-z^2}} \sqrt{(c^2-z^2)-x^2}\, dx \right\} dz \\ &= 4 \int_0^c z \left\{ \int_0^1 \left(\sqrt{c^2-z^2}\right)^2 \cdot \sqrt{1-t^2}\, dt \right\} dz \\ &= 4 \int_0^c z(c^2-z^2)\, dz \cdot \int_0^1 \sqrt{1-t^2}\, dt = c^4 \cdot \frac{\pi}{4}, \end{aligned}$$

where there are lots of similar variants.

2) **First variant.** *A symmetric argument.*
The set A is symmetric with respect to the planes $y = 0$ and $x = 0$, and the integrand x, resp. y, is an odd function. Hence,

$$\int_A x\, d\Omega = 0 \quad \text{and} \quad \int_A y\, d\Omega = 0.$$

Second variant. *Spherical coordinates.*
By insertion,

$$\begin{aligned} \int_A x\, d\Omega &= \int_0^{2\pi} \left\{ \int_0^{\frac{\pi}{2}} \left(\int_0^c r \sin\theta \cos\varphi \cdot r^2 \sin\theta\, dr \right) d\theta \right\} d\varphi \\ &= \int_0^{2\pi} \cos\varphi\, d\varphi \cdot \int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta \cdot \int_0^c r^3\, dr = [\sin\varphi]_0^{2\pi} \cdot \frac{\pi}{4} \cdot \frac{c^4}{4} = 0, \end{aligned}$$

and similarly.

Third and fourth variant. These are similar to the previous semi-polar and rectangular cases.

2 Space integrals, semi-polar coordinates

Example 2.1 *Compute in each of the following cases the given space integral over a point set A, which in semi-polar coordinates is bounded by*

$$\alpha \le \varphi \le \beta \quad \text{and} \quad (\varrho, z) \in B(\varphi).$$

One shall first from the given description of the domain of integration find α, β and $B(\varphi)$.

1) The space integral $\int_A \sqrt{x^2+y^2}\, d\Omega$, where the domain of integration A is given by

$$\sqrt{x^2+y^2} \le z \le 1.$$

2) The space integral $\int_A \ln(1+x^2+y^2)\, d\Omega$, where the domain of integration A is given by

$$\frac{1}{2}\left(x^2+x^2\right) \le z \le 2.$$

3) The space integral $\int_A (x+y^2)z\, d\Omega$, where the domain of integration A is given by

$$x^2+y^2 \le 1 \quad \text{and} \quad x^2+y^2 \le z \le \sqrt{2-x^2-y^2}.$$

4) The space integral $\int_A (x^2+y^2)\, d\Omega$, where the domain of integration A is given by

$$\frac{x^2+y^2}{a} \le z \le h.$$

5) The space integral $\int_A xy\, d\Omega$, where the domain of integration A is given by the conditions

$$x \ge 0, \quad y \ge 0 \quad \text{and} \quad \frac{x^2+y^2}{a} \le z \le h.$$

6) The space integral $\int_A xz\, d\Omega$, where the domain of integration A is given by

$$x^2+y^2 \le 2x \quad \text{and} \quad 0 \le z \le \sqrt{x^2+y^2}.$$

7) The space integral $\int_A (z^2+y^2)\, d\omega$, where the domain of integration A is given by

$$0 \le z \le h - \frac{h}{a}\sqrt{x^2+y^2}.$$

8) The space integral $\int_A (x^2+y^2)\, d\Omega$, where the domain of integration A is given by

$$x^2+y^2 \le 3 \quad \text{and} \quad 0 \le z \le \sqrt{1+x^2+y^2}.$$

9) The space integral $\int_A xy\, d\Omega$, where the domain of integration A is given by

$$x^2+y^2 \le 3 \quad \text{and} \quad 0 \le z \le \sqrt{1+x^2+y^2}.$$

10) The space integral $\int_A (x^2 z + z^3)\, d\Omega$*, where the domain of integration* A *is given by*

$$0 \le z \le \sqrt{a^2 - x^2 - y^2}.$$

11) The space integral $\int_A |y| z\, d\Omega$*, where the domain of integration* A *is given by*

$$x^2 + y^2 \le ax \quad \text{and} \quad 0 \le z \le \frac{x^2}{a}.$$

12) The space integral $\int_A xz\, d\Omega$*, where the domain of integration is one half cone of revolution of vertex* $(0,0,h)$ *and its base in the plane* $z = 0$ *given by*

$$x^2 + y^2 \le a^2 \qquad \text{for } x \ge 0.$$

13) The space integral $\int_A z\, d\Omega$*, where the domain of integration* A *is given by* $x^2 + y^2 \le (2-z)^2$ *for* $z \in [0, 2]$.

[This is also **Example 1.2.8**, *so we may compare the results. Cf. also* **Example 1.1.8**.*]*

A Space integrals in semi-polar coordinates.

D Find the interval $[\alpha, \beta]$ for φ. Describe $B(\varphi)$ in semi-polar coordinates and sketch if necessary $B(\varphi)$ in the meridian half plane. Finally, compute the space integral by using the theorem of reduction in semi-polar coordinates.

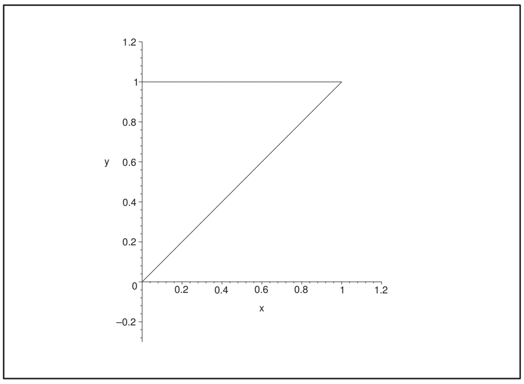

Figure 18: The meridian cut $B(\varphi)$ in **Example 2.1.1**.

I 1) Here $\varphi \in [0, 2\pi]$ and

$$B(\varphi) = \{(\varrho, z) \mid 0 \le \varrho \le 1,\ \varrho \le z \le 1\} = \{(\varrho, z) \mid 0 \le z \le 1,\ 0 \le \varrho \le z\}.$$

Then by the reduction theorem,

$$\begin{aligned}\int_A \sqrt{x^2+y^2}\, d\Omega &= \int_0^{2\pi} \left\{ \int_{B(\varphi)} \varrho \cdot \varrho \, d\varrho \, dz \right\} d\varphi \\ &= 2\pi \int_0^1 \left\{ \int_0^z \varrho^2 \, d\varrho \right\} dz = \frac{2\pi}{3} \int_0^1 z^3 \, dz = \frac{\pi}{6}.\end{aligned}$$

2) Here $\varphi \in [0, 2\pi]$, and

$$B(\varphi) = \left\{ (\varrho, z) \;\middle|\; 0 \le \varrho \le 2,\ \frac{1}{2}\,\varrho^2 \le z \le 2 \right\} = \{(\varrho, z) \mid 0 \le z \le 2,\ 0 \le \sqrt{2z}\}$$

which does not depend on φ. Then by the reduction theorem,

$$\begin{aligned}\int_A \ln(1+x^2+y^2)\, d\Omega &= \int_0^{2\pi} \left\{ \int_{B(\varphi)} \ln(1+\varrho^2) \cdot \varrho \, d\varrho \, dz \right\} d\varphi \\ &= 2\pi \int_0^2 \left\{ \int_0^{\sqrt{2z}} \ln(1+\varrho^2)\, \varrho \, d\varrho \right\} dz = 2\pi \int_0^2 \left[\frac{1}{2} \left\{ (1+\varrho^2)\ln(1+\varrho^2) - \varrho^2 \right\} \right]_{\varrho=0}^{\sqrt{2z}} dz \\ &= \pi \int_0^2 \{(1+2z)\ln(1+2z) - 2z\} dz = \frac{\pi}{2} \int_0^4 (1+t)\ln(1+t)\, dt - \pi \left[z^2\right]_{z=0}^2 \\ &= \frac{\pi}{2} \left[\frac{1}{2}(1+t)^2 \ln(1+t) - \frac{1}{4}(1+t)^2 \right]_0^4 - 4\pi = \frac{\pi}{2} \left\{ \frac{25}{2}\ln 5 - \frac{25}{4} + \frac{1}{4} \right\} - 4\pi \\ &= \pi \left\{ \frac{25}{4} \ln 5 - 7 \right\}.\end{aligned}$$

3) Here $\varphi \in [0, 2\pi]$, and $B(\varphi)$ does not depend on φ,

$$B = B(\varphi) = \{(\varrho, z) \mid 0 \le \varrho \le 1,\ \varrho^2 \le z \le \sqrt{2-\varrho^2}\}.$$

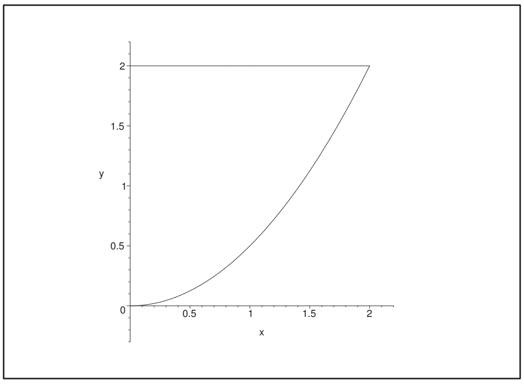

Figure 19: The meridian cut $B = B(\varphi)$ of **Example 2.1.2**.

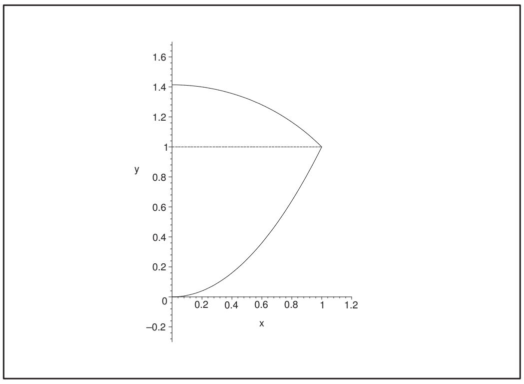

Figure 20: The meridian cut $B(\varphi)$ of **Example 2.1.3**.

It follows by the symmetry that

$$\begin{aligned}\int_A (x+y^2)z\,d\Omega &= \int_A xz\,d\Omega + \int_A y^2z\,d\Omega = 0 + \int_A y^2 z\,d\Omega \\ &= \int_0^{2\pi}\left\{\int_{B(\varphi)} \varrho^2\sin^2\varphi\cdot z\cdot\varrho\,d\varrho\,dz\right\}dz = \left\{\int_0^{2\pi}\sin^2\varphi\,d\varphi\right\}\cdot\left\{\int_B z\varrho^2\,d\varrho\,dz\right\} \\ &= \pi\int_0^1 \varrho^3\left\{\int_{\varrho^2}^{\sqrt{2-\varrho^2}} z\,dz\right\}d\varrho = \frac{\pi}{2}\int_0^1 \varrho^3\left[z^3\right]_{z=\varrho^2}^{\sqrt{2-\varrho^2}}d\varrho = \frac{\pi}{2}\int_0^1\varrho^3\left(2-\varrho^2-\varrho^4\right)d\varrho \\ &= \frac{\pi}{2}\int_0^1\left\{2\varrho^3-\varrho^5-\varrho^7\right\}d\varrho = \frac{\pi}{2}\left(\frac{1}{2}-\frac{1}{6}-\frac{1}{8}\right) = \frac{5\pi}{48}.\end{aligned}$$

4) Here $\varphi \in [0, 2\pi]$, and $B(\varphi)$ does not depend on φ,

$$B = B(\varphi) = \left\{(\varrho, z)\;\middle|\;\frac{\varrho^2}{a} \le z \le h\right\} = \{(\varrho, z) \mid 0 \le z \le h,\, 0 \le \varrho \le \sqrt{az}\}.$$

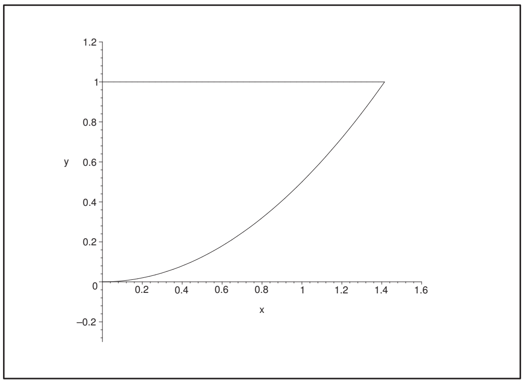

Figure 21: The meridian cut $B(\varphi)$ for $a = 2$ and $h = 1$ in **Example 2.1.4** and **Example 2.1.5**.

Then by the reduction theorem,

$$\begin{aligned}\int_A (x^2+y^2)\,d\Omega &= \int_0^{2\pi}\left\{\int_B \varrho^2\cdot\varrho\,d\varrho\,dz\right\}d\varphi = 2\pi\int_0^h\left\{\int_0^{\sqrt{az}}\varrho^3\,d\varrho\right\}dz \\ &= \frac{2\pi}{4}\int_0^h a^2z^2\,dz = \frac{\pi a^2h^3}{6}.\end{aligned}$$

5) Here $\varphi \in \left[0, \frac{\pi}{2}\right]$. Note that $B = B(\varphi)$ is the same set as in 4),

$$B = B(\varphi) = \{(\varrho, z) \mid 0 \le z \le h,\, 0 \le \varrho \le \sqrt{az}\}.$$

Then by the reduction theorem,

$$\begin{aligned}\int_A xy\,d\Omega &= \int_0^{\frac{\pi}{2}}\left\{\int_B \varrho^2\cos\varphi\cdot\sin\varphi\cdot\varrho\,d\varrho\,d<\right\}d\varphi \\ &= \int_0^{\frac{\pi}{2}}\cos\varphi\cdot\sin\varphi\,d\varphi\cdot\int_0^h\left\{\int_0^{\sqrt{az}}\varrho^3\,d\varrho\right\}dz = \left[\frac{\sin^2\varphi}{2}\right]_0^{\frac{\pi}{2}}\cdot\frac{1}{4}\int_0^h a^2z^2\,dz = \frac{a^2h^3}{24}.\end{aligned}$$

6) It follows from $x^2 + y^2 \le 2x$ that

$$\varrho \le 2\cos\varphi, \qquad \varphi \in \left[-\frac{\pi}{2}, \varphi\pi 2\right],$$

corresponding to the disc $(x-1)^2 + y^2 \le 1$ in the XY-plane. Furthermore,

$$B(\varphi) = \{(\varrho, z) \mid 0 \le \varrho \le 2\cos\varphi,\, 0 \le z \le \varrho\},$$

which depends on φ. The domain of integration A is obtained by removing the open cone $z > \sqrt{x^2+y^2}$ from the half infinite $(x-1)^2 + y^2 \le 1$.

We get by using the reduction theorem,

$$\begin{aligned}
\int_A xz\,d\Omega &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\{\int_{B(\varphi)} \varrho\cos\varphi\cdot z\cdot\varrho\,d\varrho\,dz\right\}d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\varphi\left\{\int_0^{2\cos\varphi}\varrho^2\left[\int_0^{\varrho} z\,dz\right]d\varrho\right\}d\varphi\\
&= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\{\int_0^{2\cos\varphi}\varrho^4\,d\varrho\right\}d\varphi = \frac{1}{2\cdot 5}\int_{-\frac{\pi}{2}}\frac{\pi}{2}\cos\varphi\left[\varrho^5\right]_{\varrho=0}^{2\cos\varphi}d\varphi\\
&= \frac{1}{5}\int_0^{\frac{\pi}{2}} 32\cdot\cos^6\varphi\,d\varphi = \frac{4}{5}\int_0^{\frac{\pi}{2}}(\cos 2\varphi+1)^3\,d\varphi = \frac{2}{5}\int_0^{\pi}(\cos t+1)^3\,dt\\
&= \frac{2}{5}\int_0^{\pi}\{\cos^3 t+3\cos^2 t+3\cos t+1\}dt\\
&= \frac{2}{5}\int_0^{\pi}\{(1-\sin^2 t)\cos t+\frac{3}{2}(\cos 2t+1)+3\cos t+1\}dt\\
&= \frac{2}{5}\left[-\frac{1}{3}\sin^3 t+4\sin t+\frac{3}{4}\sin 2t+\frac{5}{2}t\right]_0^{\pi} = \pi.
\end{aligned}$$

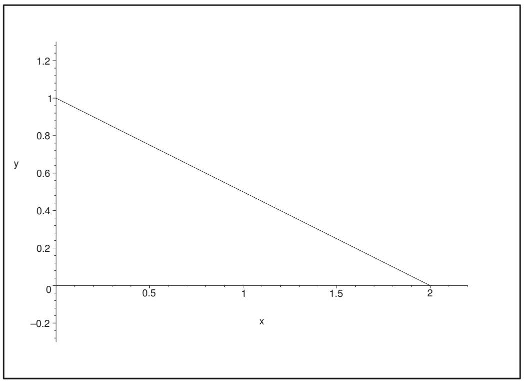

Figure 22: The meridian cut $B = B(\varphi)$ for $a = 2$ and $h = 1$ in **Example 2.1.7**.

7) Here $\varphi \in [0, 2\pi]$, and $\varrho \in [0, a]$, and $B = B(\varphi)$ does not depend on φ,

$$B = B(\varphi) = \left\{ (\varrho, z) \;\middle|\; 0 \le \varrho \le a,\, 0 \le z \le h - \frac{h}{a}\,\varrho \right\}.$$

We get by the reduction theorem,

$$\begin{aligned}
\int_A (z^2 + y^2)\, d\Omega &= \int_0^{2\pi} \left\{ \int_B z^2 \cdot \varrho \, d\varrho \, dz \right\} d\varphi + \int_0^{2\pi} \left\{ \int_B \varrho^2 \sin^2 \varphi \cdot \varrho \, d\varrho \, dz \right\} d\varphi \\
&= 2\pi \int_0^a \varrho \left\{ \int_0^{h(1-\frac{\varrho}{a})} z^2 \, dz \right\} d\varrho + \int_0^{2\pi} \sin^2 \varphi \, d\varphi \cdot \int_0^a \varrho^3 \left\{ \int_0^{h(1-\frac{\varrho}{a})} dz \right\} d\varrho \\
&= \frac{2\pi}{3} \int_0^a \varrho \cdot h^3 \left(1 - \frac{\varrho}{a}\right)^3 d\varrho + \pi \int_0^a \varrho^3 \cdot h \left(1 - \frac{\varrho}{a}\right) d\varrho \\
&= \frac{2\pi h^3}{3} \cdot a^2 \int_0^a \left\{1 - \left(1 - \frac{\varrho}{a}\right)\right\} \left(1 - \frac{\varrho}{a}\right)^3 \frac{1}{a}\, d\varrho + \pi h a^4 \int_0^a \left(\frac{\varrho}{a}\right)^3 \cdot \left(1 - \frac{\varrho}{a}\right) \frac{1}{a}\, d\varrho \\
&= \frac{2\pi h^3}{3} \cdot a^2 \int_0^1 (1-t) t^3 \, dt + \pi h a^4 \int_0^1 t^3 (1-t)\, dt = \pi h a^2 \left(\frac{2}{3}\, h^2 + a^2\right) \int_0^1 (t^3 - t^4)\, dt \\
&= \frac{\pi h a^2}{20} \left(\frac{2}{3}\, h^2 + a^2\right).
\end{aligned}$$

8) Here $\varphi \in [0, 2\pi]$ and $0 \le \varrho \le \sqrt{3}$, and

$$B = B(\varphi) = \{(\varrho, z) \mid 0 \le \varrho \le \sqrt{3},\, 0 \le z \le \sqrt{1 + \varrho^2}\},$$

which is independent of φ.

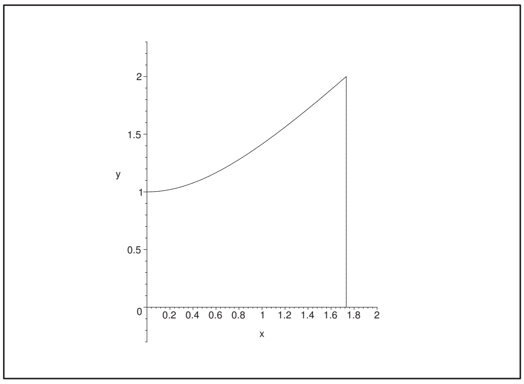

Figure 23: The meridian cut $B = B(\varphi)$ in **Example 2.1.8** and **Example 2.1.9**.

By the reduction theorem,

$$\begin{aligned}\int_A (x^2+y^2)\,d\Omega &= \int_0^{2\pi}\left\{\int_B \varrho^2\cdot\varrho\,d\varrho\,dz\right\}d\varphi = 2\pi\int_0^{\sqrt{3}}\varrho^3\left\{\int_0^{\sqrt{1+\varrho^2}}dz\right\}d\varrho\\ &= 2\pi\int_0^{\sqrt{3}}\varrho^2\sqrt{1+\varrho^2}\,d\varrho = \pi\int_0^3 t\sqrt{1+t}dt = \pi\int_0^3\left\{(1+t)^{\frac{3}{2}}-(1+t)^{\frac{1}{2}}\right\}dt\\ &= \pi\left[\frac{2}{5}(1+t)^{\frac{2}{5}}-\frac{2}{3}(1+t)^{\frac{3}{2}}\right]_0^3 = 2\pi\left(\frac{1}{5}\left\{4^{\frac{5}{2}}-1\right\}-\frac{1}{3}\left\{4^{\frac{3}{2}}-1\right\}\right)\\ &= 2\pi\left(\frac{31}{5}-\frac{7}{3}\right) = \frac{116\pi}{15}.\end{aligned}$$

9) The domain of integration is the same as in **Example 2.1.8**, so $\varphi \in [0, 2\pi]$, and

$$B = B(\varphi) = \{(\varrho, z) \mid 0 \le \varrho \le \sqrt{3},\, 0 \le z \le \sqrt{1+\varrho^2}\}.$$

Now A is symmetric with respect to e.g. the plane $y = 0$, so

$$\int_A xy\,d\Omega = 0.$$

ALTERNATIVELY we have the following calculation

$$\int_A xy\,d\Omega = \int_0^{2\pi}\left\{\int_B \varrho^2\sin\varphi\cdot\cos\varphi\cdot\varrho\,d\varrho\,dz\right\}d\varphi = \int_0^{2\pi}\sin\varphi\cdot\cos\varphi\,d\varphi\cdot\int_B\varrho^3\,d\varrho\,dz = 0,$$

where we have used that B does not depend on φ and also that

$$\int_0^{2\pi}\sin\varphi\cdot\cos\varphi\,d\varphi = \left[\frac{\sin^2\varphi}{2}\right]_0^{2\pi} = 0.$$

10) Here A is the half ball in the half space $z \ge 0$ of centrum $(0,0,0)$ and radius a, thus $\varphi \in [0, 2\pi]$, and $B(\varphi)$ does not depend on φ,

$$B = B(\varphi) = \{(\varrho, z) \mid 0 \le \varrho \le a,\, 0 \le z \le \sqrt{a^2-\varrho^2}\}.$$

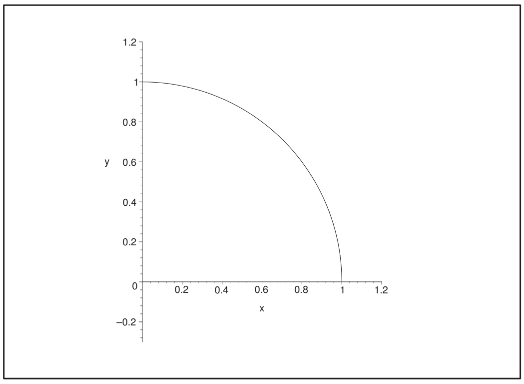

Figure 24: The meridian cut $B = B(\varphi)$ in **Example 2.1.10**.

By the reduction theorem,

$$\begin{aligned}
\int_A (x^2 z + z^3)\, d\Omega &= \int_0^{2\pi} \left\{ \int_B (\varrho^2 \cos^2\varphi \cdot z + z^3)\varrho \, d\varrho \, dz \right\} d\varphi \\
&= \int_0^{2\pi} \cos^2\varphi \, d\varphi \cdot \int_0^a \varrho^3 \left\{ \int_0^{\sqrt{a^2-\varrho^2}} z \, dz \right\} d\varrho + 2\pi \int_0^a \varrho \left\{ \int_0^{\sqrt{a^2-\varrho^2}} z^3 \, dz \right\} d\varrho \\
&= \pi \cdot 12 \int_0^a (a^2\varrho^3 - \varrho^5)\, d\varrho + \frac{2\pi}{4} \int_0^a \varrho (a^2 - \varrho^2)^2 \, d\varrho \\
&= \frac{\pi}{2} \left[\frac{a^2}{4}\varrho^4 - \frac{1}{6}\varrho^6 \right]_0^a + \frac{\pi}{2} \cdot \frac{1}{2} \int_0^{a^2} (a^2 - t)^2 \, dt \\
&= \frac{\pi}{2} \left(\frac{a^6}{4} - \frac{a^6}{6} \right) + \frac{\pi}{12} \left[-(a^2 - t)^3 \right]_0^{a^2} = \frac{\pi a^6}{24} + \frac{\pi a^6}{12} = \frac{\pi a^6}{8}.
\end{aligned}$$

11) Here $x^2 + y^2 \le ax$, hence $\varrho \le a\cos\varphi$, and $0 \le z \le \dfrac{1}{a}\varrho^2\cos^2\varphi$, and $\varphi \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2} \right]$, so

$$B(\varphi) = \left\{ (\varrho, z) \;\middle|\; 0 \le \varrho \le a\cos\varphi, \, 0 \le z \le \frac{1}{a}\varrho^2\cos^2\varphi \right\}.$$

Clearly, $B(\varphi)$ depends on φ, so we can only conclude that any meridian curve for fixed φ is a parabola in the PZ-plane, and there is no need to sketch it.

The set A is symmetric with respect to the plane $y=0$, so by the reduction theorem,

$$\begin{aligned}
\int_A |y|z\,d\Omega &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\{\int_{B(\varphi)} \varrho|\sin\varphi|\cdot z\cdot\varrho\,d\varrho\,dz\right\}d\varphi = 2\int_0^{\frac{\pi}{2}}\left\{\int_{B(\varphi)}\varrho\sin\varphi\cdot z\cdot\varrho\,d\varrho\,dz\right\}d\varphi\\
&= 2\int_0^{\frac{\pi}{2}}\left\{\int_0^{a\cos\varphi}\varrho^2\sin\varphi\left\{\int_0^{\frac{1}{a}\varrho^2\cos^2\varphi} z\,dz\right\}d\varrho\right\}d\varphi\\
&= 2\int_0^{\frac{\pi}{2}}\sin\varphi\left\{\int_0^{a\cos\varphi}\varrho^2\left[\frac{z^2}{2}\right]_{z=0}^{\frac{1}{a}\varrho^2\cos^2\varphi}d\varrho\right\}d\varphi\\
&= \int_0^{\frac{\pi}{2}}\sin\varphi\left\{\int_0^{a\cos\varphi}\frac{1}{a^2}\varrho^6\cos^4\varphi\,d\varrho\right\}d\varphi = \frac{1}{a^2}\int_0^{\frac{\pi}{2}}\sin\varphi\cdot\cos^4\varphi\left\{\int_0^{a\cos\varphi}\varrho^6\,d\varrho\right\}d\varphi\\
&= \frac{1}{7a^2}\int_0^{\frac{\pi}{2}}\sin\varphi\cdot\cos^4\varphi\cdot a^7\cos^7\varphi\,d\varphi = \frac{a^5}{7}\left[-\frac{\cos^{12}\varphi}{12}\right]_0^{\frac{\pi}{2}} = \frac{a^5}{84}.
\end{aligned}$$

12) Here $\varphi\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, and we have for any fixes φ that

$$B(\varphi)=\left\{(\varrho,z)\;\middle|\;0\le x\le h,\,0\le\varrho\le a\left(1-\frac{z}{h}\right)\right\}.$$

The meridian cut does not depend on $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Notice that it is identical with the meridian cut in **Example 2.1.7**.

We get by the reduction theorem in semi-polar coordinates,

$$\begin{aligned}
\int_A xz\,d\Omega &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ \int\!\!\int_{B(\varphi)} \varrho\cos\varphi \cdot z \cdot \varrho\,d\varrho\,dz \right\} d\varrho \\
&= \int_{-\frac{\pi}{2}}^{\frac{\varphi}{2}} \cos\varphi \left\{ \int_0^h z \left(\int_0^{a\left(1-\frac{z}{h}\right)} \varrho^2\,d\varrho \right) dz \right\} d\varphi = [\sin\varphi]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^h z \left[\frac{1}{3}\,\varrho^3 \right]_0^{a\left(1-\frac{z}{h}\right)} dz \\
&= 2 \cdot \frac{a^3}{3} \int_0^h z \left(1 - \frac{z}{h}\right)^3 dz = \frac{2}{3}\,a^3 \int_0^h z \left(1 - \frac{3}{h}\,z + \frac{3}{h^2}\,z^2 - \frac{1}{h^3}\,z^3 \right) dz \\
&= \frac{2}{3}\,a^3 \int_0^h \left(z - \frac{3}{h}\,z^2 + \frac{3}{h^2}\,z^3 - \frac{1}{h^3}\,z^4 \right) dz = \frac{2}{3}\,a^3 \left[\frac{1}{2}\,z^2 - \frac{1}{h}\,z^3 + \frac{3}{4h^2}\,z^4 - \frac{1}{5h^2}\,z^5 \right]_0^h \\
&= \frac{2}{3}\,a^3h^2 \left(\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5} \right) = \frac{2}{3} \left(\frac{1}{4} - \frac{1}{5} \right) a^3h^2 = \frac{1}{30}\,a^3h^2.
\end{aligned}$$

13) In this case we integrate over the same set as in **Example 1.1.8**. Then by the reduction theorem in semi-polar coordinates followed by the change of variables $u = 2 - z$,

$$\begin{aligned}
\int_A z\,d\Omega &= \int_0^2 z \cdot \pi(2-z)^2\,dz = \pi \int_0^2 (2-u)u^2\,du = \pi \int_0^2 (2u^2 - u^3)\,du \\
&= \pi \left[\frac{2}{3}\,u^3 - \frac{1}{4}\,u^4 \right]_0^2 = \pi \left[\frac{16}{3} - \frac{16}{4} \right] = \frac{16}{12}\,\pi = \frac{4\pi}{3}.
\end{aligned}$$

3 Space integrals, spherical coordinates

Example 3.1 *Calculate in each of the following cases the given space integral over a point set A, which in spherical coordinates is bounded by*

$$\alpha \le \varphi \le \beta \quad \text{and} \quad (r,\theta) \in B^*(\varphi);$$

1) The space integral $\int_A \sqrt{x^2+y^2}\, d\Omega$, *where the domain of integration A is given by*

$$x^2+y^2+z^2 \le 2.$$

2) The space integral $\int_A (x^2+y^2+z^2)^2\, d\Omega$, *where the domain of integration A is given by*

$$x^2+y^2+z^2 \le 1.$$

3) The space integral $\int_A xyz\, d\Omega$, *where the domain of integration A is given by*

$$x^2+y^2+z^2 \le 1, \quad x \ge 0, \quad y \ge 0, \quad z \ge 0.$$

4) The space integral $\int_A (x^2+y^2+z^2)^{-\frac{3}{2}}\, d\Omega$, *where the domain of integration A is given by*

$$a^2 \le x^2+y^2+z^2 \le b^2, \qquad \text{where } b > a.$$

5) The space integral $\int_A (x^2z+z^3)\, d\Omega$, *where the domain of integration A is given by*

$$x^2+y^2+z^2 \le a^2 \quad \text{and} \quad z \ge 0.$$

6) The space integral $\int_A \frac{y}{z^2}\, d\Omega$, *where the domain of integration A is given by*

$$x^2+y^2+z^2 \le (2a)^2, \quad a \le z, \quad 0 \le y \le x.$$

A Space integrals in spherical coordinates.

D Identify the point set. Sketch if necessary the meridian cut. Finally, compute the space integral by reduction in spherical coordinates.

I 1) It is obvious that A is a conic slice of the ball of centrum $(0,0,0)$ and radius $\sqrt{2}$. Thus $0 \le \varphi \le 2\pi$, and the meridian cut

$$B^* = B^*(\varphi) = \left\{ (r,\theta) \;\middle|\; 0 \le r \le \sqrt{2},\, 0 \le \theta \le \frac{\pi}{4} \right\}$$

does not depend on φ Then by the reduction theorem in spherical coordinates,

$$\begin{aligned}
\int_A \sqrt{x^2+y^2}\, d\Omega &= \int_0^{2\pi} \left\{ \int_{N^*(\varphi)} r\sin\theta \cdot r^2 \sin\theta\, dr\, d\theta \right\} d\varphi = 2\pi \int_0^{\sqrt{2}} r^3\, dr \cdot \int_0^{\frac{\pi}{4}} \sin^2\theta\, d\theta \\
&= 2\pi \left[\frac{r^4}{4} \right]_0^{\sqrt{2}} \int_0^{\frac{\pi}{4}} \frac{1-\cos 2\theta}{2}\, d\theta = 2\pi \cdot \frac{4}{4} \cdot \frac{1}{2} \left[\theta - \frac{1}{2}\sin 2\theta \right]_0^{\frac{\pi}{4}} \\
&= \pi \left(\frac{\pi}{4} - \frac{1}{2} \right) = \frac{\pi^2}{4} - \frac{\pi}{2}.
\end{aligned}$$

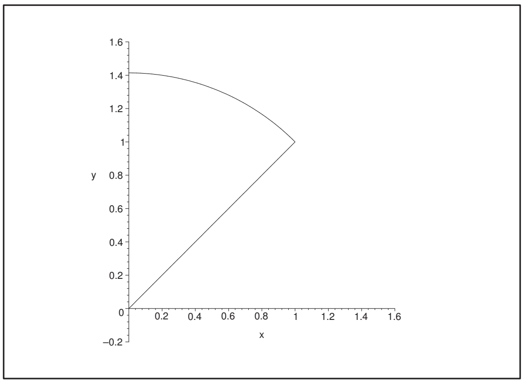

Figure 25: The meridian cut B^* in **Example 3.1.1**.

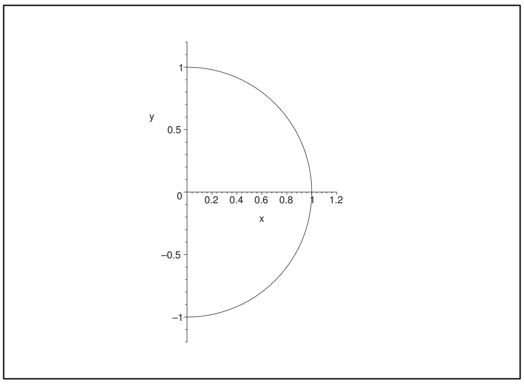

Figure 26: The meridian cut B^* in **Example 3.1.2**.

2) The set A is the unit ball, so $\varphi \in [0, 2\pi]$, and $B^* = B^*(\varphi)$ is the unit half circle in the right half plane which does not depend on φ,

$$B^* = B^*(\varphi) = \{(r, \theta) \mid 0 \le r \le 1,\, 0 \le \theta \le \pi\}.$$

Then by the reduction theorem in spherical coordinates,

$$\int_A (x^2+y^2+z^2)^2 \, d\Omega = 2\pi \int_{B^*} r^4 \cdot r^2 \, \sin\theta \, dr \, d\theta = 2\pi \int_0^1 r^6 \cdot \int_0^\pi \sin\theta \, d\theta = \frac{4\pi}{7}.$$

3) The domain of integration is that part of the unit ball which lies in the first octant, thus $0 \le \varphi \le \dfrac{\pi}{2}$ and

$$B^*(\varphi) = \left\{ (r, \theta) \;\middle|\; 0 \le \theta \le \frac{\pi}{2} \right\} \qquad \text{for } 0 \le \varphi \le \frac{\pi}{2}.$$

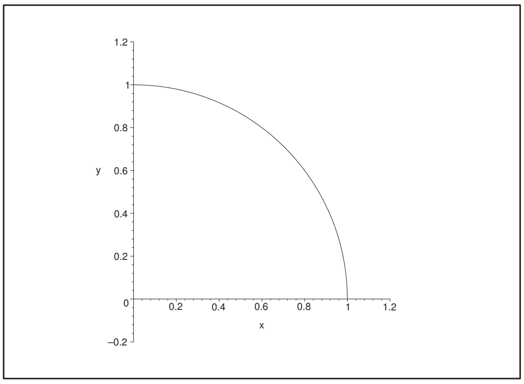

Figure 27: The meridian cut $B^*(\varphi)$, $\varphi \in \left[0, \frac{\pi}{2}\right]$ in **Example 3.1.3**.

By the reduction theorem in spherical coordinates,

$$\begin{aligned}\int_A xyz\, d\Omega &= \int_0^{\frac{\pi}{2}} \left\{ \int_{B^*} r^3 \sin^2\theta \cos\theta \cdot \sin\varphi\, \cos\varphi \cdot r^2 \sin\theta\, dr\, d\theta \right\} d\varphi \\ &= \int_0^{\frac{\pi}{2}} \sin\varphi \cdot \cos\varphi\, d\varphi \cdot \int_0^1 r^5\, dr \cdot \int_0^{\frac{\pi}{2}} \sin^3\theta\, \cos\theta\, d\theta \\ &= \left[\frac{1}{2}\sin^2\varphi\right]_0^{\frac{\pi}{2}} \cdot \left[\frac{r^2}{6}\right]_0^1 \cdot \left[\frac{1}{4}\sin^4\theta\right]_0^{\frac{\pi}{2}} = \frac{1}{2}\cdot\frac{1}{6}\cdot\frac{1}{4} = \frac{1}{48}.\end{aligned}$$

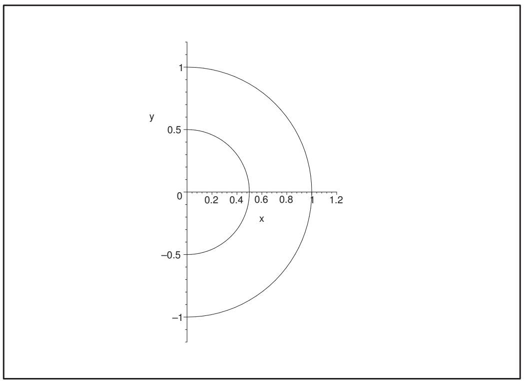

Figure 28: The meridian cut B^* in **Example 3.1.4** for $a = \frac{1}{2}$ and $b = 1$.

4) Here A is a shell, so $\varphi \in [0, 2\pi]$, and

$$B^* = B^*(\varphi) = \{(r,\theta) \mid a \le r \le b,\ 0 \le \theta \le \pi\}$$

does not depend on φ.

By the reduction theorem in spherical coordinates,

$$\int_A (x^2+y^2+z^2)^{-\frac{3}{2}}\, d\Omega = \int_0^{2\pi} \left\{ \int_{B^*} r^{-3}\, r^2 \sin\theta \, dr\, d\theta \right\} d\varphi$$

$$= 2\pi \int_1^b \frac{1}{r}\, dr \cdot \int_0^{\pi} \sin\theta \, d\theta = 2\pi [\ln r]_a^b \cdot [-\cos\theta]_0^{\pi} = 4\pi \ln\left(\frac{b}{a}\right).$$

5) Here A is that part of the ball of centrum $(0,0,0)$ and radius a, which lies in the upper half space, thus $0 \le \varphi \le 2\pi$, and

$$B^* = B^*(\varphi) = \left\{ (r,\theta) \;\middle|\; 0 \le r \le a,\, 0 \le \theta \le \frac{\pi}{2} \right\}.$$

By the reduction theorem in spherical coordinates,

$$\begin{aligned}
\int_A (x^2z+z^3)\,d\Omega &= \int_0^{2\pi}\left\{\int_{B^*}(r^2\sin^2\theta\cos^2\varphi\, r\cos\theta + r^3\cos^3\theta)r^2\sin\theta\,dr\,d\theta\right\}d\varphi \\
&= \int_0^{2\pi}\left\{\int_0^a r^5\left\{\int_0^{\frac{\pi}{2}}(\cos^2\varphi\,\sin^2\theta\,\cos\theta+\cos^3\theta)\sin\theta\,d\theta\right\}dr\right\}d\varphi \\
&= \left[\frac{r^6}{6}\right]_0^a\int_0^{2\pi}\left\{\int_0^{\frac{\pi}{2}}\{\cos^2\varphi(\cos\theta-\cos^3\theta)+\cos^3\theta\}\sin\theta\,d\theta\right\}d\varphi \\
&= \frac{a^6}{6}\int_0^{2\pi}\left\{\int_0^{\frac{\pi}{2}}\{\cos^2\varphi\,\cos\theta+\sin^2\varphi\,\cos^3\theta\}\sin\theta\,d\theta\right\}d\varphi \\
&= \frac{a^6}{5}\int_0^{2\pi}\left[-\cos^2\varphi\cdot\frac{1}{2}\,\cos^2\theta-\sin^2\varphi\cdot\frac{1}{4}\,\cos^4\theta\right]_{\theta=0}^{\frac{\pi}{2}}d\varphi \\
&= \frac{a^6}{24}\int_0^{2\pi}\{2\cos^2\varphi+\sin^2\varphi\}d\varphi = \frac{a^6}{24}\int_0^{2\pi}\left\{\frac{3}{2}+\frac{1}{2}\,\cos 2\varphi\right\}d\varphi \\
&= \frac{a^6}{24}\cdot\frac{3}{2}\cdot 2\pi = \frac{a^6\pi}{8}.
\end{aligned}$$

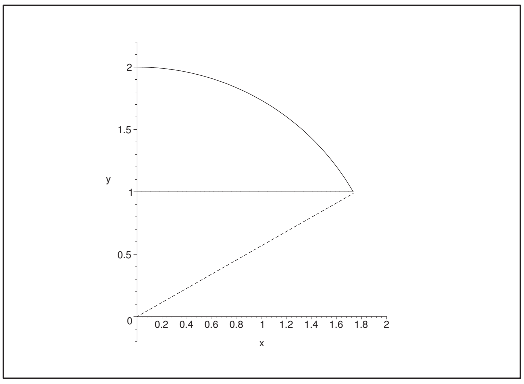

Figure 29: The meridian cut $B^*(\varphi)$ for $\varphi\in\left[0,\frac{\pi}{4}\right]$ and $a=1$ in **Example 3.1.6**.

6) The domain of integration is in spherical coordinates described by $0\le\varphi\le\frac{\pi}{4}$ (from the request $0\le y\le x$) and

$$\begin{aligned}
B^*(\varphi) &= \left\{(r,\theta)\;\middle|\;0\le\theta\le\frac{\pi}{3},\,a\le r\cos\theta,\,r\le 2a\right\} \\
&= \left\{(r,\theta)\;\middle|\;0\le\theta\le\frac{\pi}{3},\,\frac{a}{\cos\theta}\le r\le 2a\right\},
\end{aligned}$$

for $\varphi\in\left[0,\frac{\pi}{4}\right]$. We see that $B^*=B^*(\varphi)$ does not change in this φ-interval, hence by a reduction in spherical coordinates,

$$\begin{aligned}
\int_A \frac{y}{z^2}\,d\Omega &= \int_0^{\frac{\pi}{4}} \left\{ \int_{B^*} \frac{r\sin\theta\,\sin\varphi}{r^2\cos^2\theta}\cdot r^2\sin\theta\,dr\,d\theta\right\} d\varphi \\
&= \int_0^{\frac{\pi}{4}} \left\{ \int_0^{\frac{\pi}{3}} \left(\int_{\frac{a}{\cos\theta}}^{2a} r\cdot\frac{\sin^2\theta}{\cos^2\theta}\cdot\sin\varphi\,dr\right) d\theta\right\} d\varphi \\
&= [-\cos\varphi]_0^{\frac{\pi}{4}}\cdot\int_0^{\frac{\pi}{3}} \frac{\sin^2\theta}{\cos^2\theta}\left[\frac{1}{2}r^2\right]_{\frac{a}{\cos\theta}}^{2a} d\theta \\
&= \left(1-\frac{1}{\sqrt{2}}\right)\int_0^{\frac{\pi}{3}} \frac{\sin^2\theta}{\cos^2\theta}\left(2a^2-\frac{a^2}{2}\,\frac{1}{\cos^2\theta}\right) d\theta \\
&= \left(1-\frac{1}{\sqrt{2}}\right)a^2\left\{2\int_0^{\frac{\pi}{3}}\frac{1-\cos^2\theta}{\cos^2\theta}\,d\theta-\frac{1}{2}\int_0^{\frac{\pi}{3}}\tan^2\theta\cdot\frac{1}{\cos^2\theta}\,d\theta\right\} \\
&= \left(1-\frac{1}{\sqrt{2}}\right)a^2\left\{2[\tan\theta-\theta]_0^{\frac{\pi}{3}}-\frac{1}{2}\left[\frac{1}{3}\tan^3\theta\right]_0^{\frac{\pi}{3}}\right\} \\
&= \left(1-\frac{1}{\sqrt{2}}\right)\left\{2\left(\sqrt{3}-\frac{\pi}{3}\right)-\frac{1}{6}\cdot 3\sqrt{3}\right\} = \left(1-\frac{\sqrt{2}}{2}\right)\left(\frac{3}{2}\sqrt{3}-\frac{2}{3}\pi\right)a^2.
\end{aligned}$$

Example 3.2 *Let a denote a positive constant. Let K_0 denote the closed half ball of centrum $(0,0,0)$, of radius $2a$, and where $z \geq 0$. Finally, let lad K_1 denote the open ball of centrum $(0,0,a)$ and radius a. We define a closed body of revolution A by removing K_1 from K_0. Thus $A = K_0 \setminus K_1$. Let B denote a meridian cut in A.*

1) Sketch B, and explain why A in spherical coordinates (r,θ,φ) is given by

$$\varphi \in [0,2\pi], \qquad \theta \in \left[0,\frac{\pi}{2}\right], \qquad r \in [2a\,\cos\theta, 2a].$$

2) Compute the space integral $\int_A z^2\,d\Omega$.

A Space integral in spherical coordinates.

D Analyze geometrically the meridian half plane (add a line perpendicular to the radius vector). Then use the spherical reduction of the space integral.

I 1) The figure shows that we have a rectangular triangle with the hypothenuse of length $2a$ along the Z-axis and the angle θ between radius vector and the Z-axis. Then a geometrical consideration shows that the distance from origo to the intersection point with the circle of radius a and centrum $(0,a)$ is give by $2a\,\cos\theta$. This gives us the lower limit for r, thus

$$r \in [2a\,\cos\theta, 2a].$$

The domains of the other coordinates are obvious.

2) The integrand is written in spherical coordinates in the following way,

$$f(x,y,z) = z^2 = r^2\cos^2\theta.$$

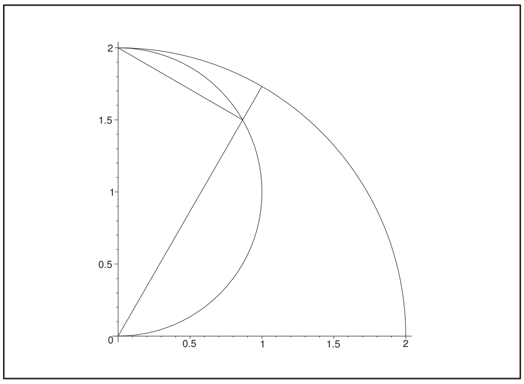

Figure 30: The meridian cut for $a = 1$ with a radius vector and a perpendicular line.

Then by the reduction theorem for space integrals in spherical coordinates,

$$\begin{aligned}\int_A z^2\,d\Omega &= 2\pi\int_0^{\frac{\pi}{2}}\left\{\int_{2a\cos\theta}^{2a} r^2\cos^2\theta\cdot r^2\sin\theta\,dr\right\}d\theta = 2\pi\int_0^{\frac{\pi}{2}}\cos^2\theta\sin\theta\cdot\left[\frac{r^5}{5}\right]_{2a\cos\theta}^{2a}d\theta\\ &= \frac{64\pi a^5}{5}\int_0^{\frac{\pi}{2}}\left\{\cos^2-\cos^7\theta\right\}\sin\theta\,d\theta = \frac{64\pi a^5}{5}\left[\frac{1}{8}\cos^8\theta-\frac{1}{3}\cos^3\theta\right]_0^{\frac{\pi}{2}}\\ &= \frac{64\pi a^5}{5}\cdot\frac{8-3}{8\cdot 3} = \frac{8\pi a^5}{3}.\end{aligned}$$

Example 3.3 *Let*

$$A=\{(x,y,z)\in\mathbb{R}^3\mid z\geq 0,\ x^2+y^2+z^2\leq 4,\ \frac{1}{3}(x^2+y^2)\leq z^2\leq 3(x^2+y^2)\}.$$

1) Sketch the curve of the intersection with the (x,z)-plane.

2) Compute the space integral

$$\int_A z\,d\Omega.$$

(It is an advantage here to use spherical coordinates).

A Space integral in spherical coordinates.

D Follow the guidelines of the text.

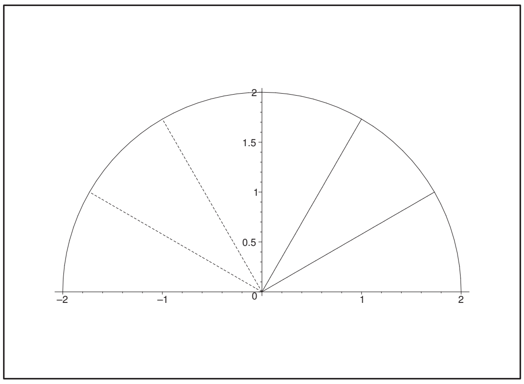

Figure 31: The intersection curve with the (x,z)-plane. It follows from the symmetry that we are only interested in the sector of the first quadrant.

I 1) It $y=0$, then we get the limitations $z\geq 0$, $x^2+z^2\leq 2^2$ and $\dfrac{1}{3}\,x^2\leq z^2\leq 3x^2$, thus

$$\frac{|x|}{\sqrt{3}}\leq z\leq\sqrt{3}\,|x|.$$

The intersection curve is given in spherical coordinates (r,θ) by

$$\left\{(r\theta)\;\middle|\;r\in[0,2],\,\theta\in\left[\frac{\pi}{6},\frac{\pi}{3}\right]\cup\left[-\frac{\pi}{3},-\frac{\pi}{6}\right]\right\},$$

where θ is measured from the Z-axis and positive towards the X-axis.

2) The space integral is then computed by reduction in spherical coordinates,

$$\begin{aligned}\int_A z\,d\Omega &= \int_0^{2\pi}\left\{\int_0^2\left\{\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} r\,\cos\theta\cdot r^2\sin\theta\,d\theta\right\}dr\right\}d\varphi\\ &= 2\pi\int_0^2 r^3\,dr\cdot\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\sin\theta\cdot\cos\theta\,d\theta=2\pi\left[\frac{r^4}{4}\right]_0^2\left[\frac{\sin^2\theta}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}\\ &= 2\pi\cdot\frac{16}{4}\cdot\frac{1}{2}\left(\frac{3}{4}-\frac{1}{4}\right)=2\pi\cdot 4\cdot\frac{1}{2}\cdot\frac{1}{2}=2\pi.\end{aligned}$$

Example 3.4 *Let a and c be positive constants, and let A denote the half shell given by the inequalities*

$$a^2 \le x^2 + y^2 + z^2 \le 4a^2, \qquad z \ge 0,$$

Compute the space integral

$$\int_A \frac{z}{c^2 + x^2 + y^2 + z^2}\, d\Omega.$$

A Space integral.

D I have here found four variants:

1) Reduction in spherical coordinates.
2) Reduction in semi-polar coordinates.
3) Reduction by the slicing method.
4) Reduction in rectangular coordinate.

These methods are here numbered according to their increasing difficulty. The fourth variant is possible, but it is not worth here to produce all the steps involved, because the method cannot be recommended i this particular case.

I First variant. *Spherical coordinates.*

The set A is described in spherical coordinates by

$$\left\{(r, \varphi, \theta) \;\middle|\; r \in [a, 2a],\, \varphi \in [0, 2\pi],\, \theta \in \left[0, \frac{\pi}{2}\right]\right\},$$

hence by the reduction of the space integral,

$$\begin{aligned}
\int_A \frac{z}{c^2 + x^2 + y^2 + z^2}\, d\Omega &= \int_0^{2\pi} \left\{ \int_0^{\frac{\pi}{2}} \left(\int_a^{2a} \frac{r\cos\theta}{c^2 + r^2} \cdot r^2 \sin\theta\, dr \right) d\theta \right\} d\varphi \\
&= 2\pi \int_0^{\frac{\pi}{2}} \cos\theta\, \sin\theta\, d\theta \cdot \int_a^{2a} \frac{r^2}{c^2 + r^2} \cdot r\, dr \qquad [t = r^2] \\
&= 2\pi \left[\frac{\sin^2\theta}{2} \right]_0^{\frac{\pi}{2}} \cdot \int_{a^2}^{4a^2} \frac{t + c^2 - c^2}{c^2 + t} \cdot \frac{1}{2}\, dt \\
&= \frac{\pi}{2} \int_{a^2}^{4a^2} \left\{ 1 - \frac{c^2}{c^2 + t} \right\} dt = \frac{\pi}{2} \left[t - c^2 \ln\left(c^2 + t\right) \right]_{t=a^2}^{4a^2} = \frac{\pi}{2} \left\{ 3a^2 - c^2 \ln\left(\frac{4a^2 + c^2}{a^2 + c^2} \right) \right\}.
\end{aligned}$$

2. variant. *Semi-polar coordinates.*

We must here split the investigation into two according to whether $\varrho \in [0, a[$ or $\varrho \in [a, 2a]$, cf. the figure.

That part A_1 of A, which is given by $\varrho \in [0, a[$, is described in semi-polar coordinates by

$$\{(\varrho, \varphi, z) \mid \varrho \in [0, a[,\, \varphi \in [0, 2\pi],\, \sqrt{a^2 - \varrho^2} \le z \le \sqrt{4a^2 - \varrho^2}\}.$$

That part A_2 of A, which is given by $\varrho \in [a, 2a]$, is described in semi-polar coordinates by

$$\{(\varrho, \varphi, z) \mid \varrho \in [a, 2a],\, \varphi \in [0, 2\pi],\, 0 \le z \le \sqrt{4a^2 - \varrho^2}\}.$$

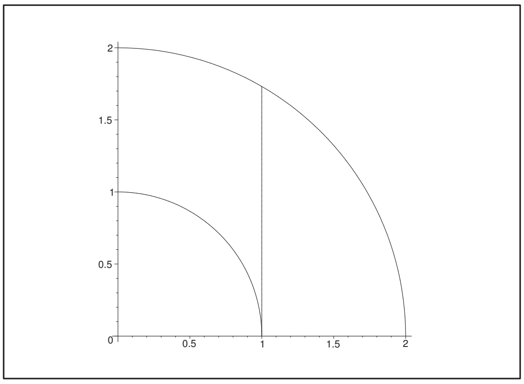

Figure 32: The meridian cut for $a = 1$ with the line $x = a = 1$.

Then by reduction in semi-polar coordinates,

$$\int_A \frac{z}{c^2+x^2+y^2+z^2}\,d\Omega = \int_{A_1} \frac{z}{c^2+x^2+y^2+z^2}\,d\Omega + \int_{A_2} \frac{z}{c^2+x^2+y^2+z^2}\,d\Omega$$

$$= \int_0^{2\pi}\left\{\int_0^a\left(\int_{\sqrt{a^2-\varrho^2}}^{\sqrt{4a^2-\varrho^2}} \frac{z}{c^2+\varrho^2+z^2}\,\varrho\,dz\right)d\varrho\right\}d\varphi$$

$$+\int_0^{2\pi}\left\{\int_1^{2a}\left(\int_0^{\sqrt{4a^2-\varrho^2}} \frac{z}{c^2+\varrho^2+z^2}\,\varrho\,dz\right)d\varrho\right\}d\varphi$$

$$= 2\pi\int_0^a\left[\frac{1}{2}\,\ln\left(c^2+\varrho^2+z^2\right)\right]_{z=\sqrt{a^2-\varrho^2}}^{\sqrt{4a^2-\varrho^2}}\varrho\,d\varrho + 2\pi\int_a^{2a}\left[\frac{1}{2}\,\ln\left(c^2+\varrho^2+z^2\right)\right]_{z=0}^{\sqrt{4a^2-\varrho}}\varrho\,d\varrho$$

$$= \pi\int_0^a\left\{\ln\left(c^2+4a^2\right)-\ln\left(c^2+a^2\right)\right\}\varrho\,d\varrho + \pi\int_a^{2a}\left\{\ln\left(c^2+4a^2\right)-\ln\left(c^2+\varrho^2\right)\right\}\varrho\,d\varrho,$$

thus

$$\int_A \frac{z}{c^2+x^2+y^2+z^2}\,d\Omega$$

$$= \pi\left\{\ln\left(c^2+4a^2\right)-\ln\left(c^2+a^2\right)\right\}\cdot\frac{a^2}{2} + \pi\ln\left(c^2+4a^2\right)\cdot\frac{1}{2}\left\{4a^2-a^2\right\},$$

$$-\frac{\pi}{2}\int_{a^1}^{4a^2}\ln\left(c^2+t\right)\,dt \qquad\qquad (\text{ved } t=\varrho^2)$$

$$= \frac{\pi}{2}\cdot 4a^2\ln\left(c^2+4a^2\right) - \frac{\pi}{2}\,a^2\,\ln\left(c^2+a^2\right) - \frac{\pi}{2}\,\left[\left(c^2+t\right)\,\ln\left(c^2+t\right)-t\right]_{t=a^2}^{4a^2}$$

$$= \frac{\pi}{2}\cdot 4a^2\,\ln\left(c^2+4a^2\right) - \frac{\pi}{2}\,a^2\,\ln\left(c^2+a^2\right) - \frac{\pi}{2}\left(c^2+4a^2\right)\ln\left(c^2+4a^2\right)$$

$$+\frac{\pi}{2}\cdot 4a^2 + \frac{\pi}{2}\,\left(c^2+a^2\right)\,\ln\left(c^2+a^2\right) - \frac{\pi}{2}\cdot a^2$$

$$= \frac{\pi}{2}\cdot\left\{3a^2 - c^2\ln\left(\frac{c^2+4a^2}{c^2+a^2}\right)\right\}.$$

Third variant. *The slicing method.*

The plane at height $z = [0, a[$ intersects A in an annulus $B(z)$, which is described in polar coordinates by

$$\left\{(\varrho, \varphi) \mid \varphi \in [0, 2\pi],\ \sqrt{a^2 - z^2} \le \varrho \le \sqrt{4a^2 - z^2}\right\}.$$

The plane at height $z \in [a, 2a]$ intersects A in a disc $B(z)$, which is described in polar coordinates by

$$\left\{(\varrho, \varphi) \mid \varphi \in [0, 2\pi],\ 0 \le \varrho \le \sqrt{4a^2 - z^2}\right\}.$$

If we first integrate over $B(z)$ and then with respect to z, we get the following reduction,

$$\begin{aligned}
&\int_A \frac{z}{c^2+x^2+y^2+z^2}\,d\Omega \\
&\quad= \int_0^a \left\{\int_{B(z)} \frac{z}{c^2+x^2+y^2+z^2}\,dS\right\}dz + \int_a^{2a}\left\{\int_{B(z)} \frac{z}{c^2+x^2+y^2+z^2}\,dS\right\}dz \\
&\quad= \int_0^a \left\{\int_0^{2\pi}\left(\int_{\sqrt{a^2-z^2}}^{\sqrt{4a^2-z^2}} \frac{z}{c^2+z^2+\varrho^2}\,\varrho\,d\varrho\right)d\varphi\right\}dz \\
&\qquad\qquad + \int_a^{2a}\left\{\int_0^{2\pi}\left(\int_0^{\sqrt{4a^2-z^2}} \frac{z}{c^2+z^2+\varrho^2}\,\varrho\,d\varrho\right)d\varphi\right\}dz,
\end{aligned}$$

hence

$$\begin{aligned}
&\int_A \frac{z}{c^2+x^2+y^2+z^2}\,d\Omega \\
&\quad= 2\pi\int_0^a \left[\frac{1}{2}\ln\left(c^2+z^2+\varrho^2\right)\right]_{\varrho=\sqrt{a^2-z^2}}^{\sqrt{4a^2-z^2}} z\,dz + 2\pi\int_a^{2a}\left[\frac{1}{2}\ln\left(c^2+z^2+\varrho^2\right)\right]_{\varrho=0}^{\sqrt{4a^2-z^2}} z\,dz \\
&\quad= \pi\int_0^a\left\{\ln\left(c^2+4a^2\right)-\ln\left(c^2+a^2\right)\right\}z\,dz + \pi\int_a^{2a}\left\{\ln\left(c^2+4a^2\right)-\ln\left(c^2+z^2\right)\right\}\,dz \\
&\quad= \pi\cdot\frac{a^2}{2}\,\ln\left(\frac{c^2+4a^2}{c^2+a^2}\right)+\pi\,\ln\left(c^2+4a^2\right)\cdot\left[\frac{z^2}{2}\right]_a^{2a} - \frac{\pi}{2}\int_{a^2}^{4a^2}\ln\left(c^2+t\right)\,dt \qquad (\text{“}t=z^2\text{”}) \\
&\quad= \frac{\pi}{2}\cdot a^2\ln\left(\frac{c^2+4a^2}{c^2+a^2}\right)+\frac{\pi}{2}\cdot 3a^2\ln\left(c^2+4a^2\right)-\frac{\pi}{2}\,\left[\left(c^2+t\right)\ln\left(c^2+t\right)-t\right]_{t=a^2}^{4a^2} \\
&\quad= 2\pi a^2\ln\left(c^2+4a^2\right)-\frac{\pi}{2}\,a^2\ln\left(c^2+a^2\right) \\
&\qquad\qquad -\frac{\pi}{2}(c^2+4a^2)\ln(c^2+4a^2)+\frac{\pi}{2}(c^2+^2)\ln(c^2+a^2)+\frac{\pi}{2}\cdot 3a^2 \\
&\quad= \frac{\pi}{2}\left\{3a^2-c^2\ln\left(\frac{c^2+4a^2}{c^2+a^2}\right)\right\}.
\end{aligned}$$

Fourth variant. *Rectangular coordinates.*

This is a very impossible variant, which I have only been through once. The computations here are only sketchy just to scare people away, because it cannot be recommended.

Let

$$A_0 = \{(x,y,z) \mid a^2 \le x^2+y^2+z^2 \le 4a^2,\ x\ge 0,\ y\ge 0,\ z\ge 0\}$$

be that part of A, which lies in the first octant. Then by an argument of symmetry on the integrand we conclude that

$$\int_A \frac{z}{c^2+x^2+y^2+z^2}\,d\Omega = 4\int_{A_0}\frac{z}{c^2+x^2+y^2+z^2}\,d\Omega.$$

When $x\in[0,a[$ is fixed, the corresponding plane intersects the set A_0 in a domain $B(x)$, which is given in rectangular coordinates by

$$\begin{aligned}
&\{(y,z)\mid y\in[0,\sqrt{a^2-x^2}],\ \sqrt{a^2-x^2-y^2}\le z\le\sqrt{4a^2-x^2-y^2}\} \\
&\qquad\cup\{(y,z)\mid y\in]\sqrt{a^2-x^2},\sqrt{4a^2-x^2}],\ 0\le z\le\sqrt{4a^2-x^2-y^2}\}.
\end{aligned}$$

Remark. We see that the description in polar coordinates would be easier here, but I shall here demonstrate how bad things can be if one only uses rectangular coordinates. $\Diamond$

Similarly, A_0 is cut for $x \in [a, 2a]$ into a quarter disc

$$\{(y,z) \mid y \in [0, \sqrt{4a^2 - x^2}],\, 0 \le z \le \sqrt{4a^2 - x^2 - y^2}\}.$$

Then by reduction in rectangular coordinates

$$\begin{aligned}
\int_A \frac{z}{c^2+x^2+y^2+z^2}\,d\Omega &= 4\int_{A_0} \frac{z}{c^2+x^2+y^2+z^2}\,d\Omega \\
&= 4\int_0^a \left\{ \int_0^{\sqrt{a^2-x^2}} \left(\int_{\sqrt{a^2-x^2-y^2}}^{\sqrt{4a^2-x^2-y^2}} \frac{z}{c^2+x^2+y^2+z^2}\,dz \right) dy \right\} dx \\
&\quad +4\int_0^a \left\{ \int_{\sqrt{a^2-x^2}}^{\sqrt{4a^2-x^2}} \left(\int_0^{\sqrt{4a^2-x^2-y^2}} \frac{z}{c^2+x^2+y^2+z^2}\,dz \right) dy \right\} dx \\
&\quad +4\int_a^{2a} \left\{ \int_0^{\sqrt{4a^2-x^2}} \left(\int_0^{\sqrt{4a^2-x^2-y^2}} \frac{z}{c^2+x^2+y^2+z^2}\,dz \right) dy \right\} dx \\
&= 2\int_0^a \left\{ \int_0^{\sqrt{a^2-x^2}} \left\{ \ln\left(c^2+4a^2\right) - \ln\left(c^2+a^2\right) \right\} dy \right\} dx \\
&\quad +2\int_0^a \left\{ \int_{\sqrt{a^2-x^2}}^{\sqrt{4a^2-x^2}} \left\{ \ln\left(c^2+4a^2\right) - \ln\left(c^2+x^2+y^2\right) \right\} dy \right\} dx \\
&\quad +2\int_a^{2a} \left\{ \int_0^{\sqrt{4a^2-x^2}} \left\{ \ln\left(c^2+4a^2\right) - \ln\left(c^2+x^2+y^2\right) \right\} dy \right\} dx.
\end{aligned}$$

The former of these integrals is easy to compute, because it is a constant integrated over a quarter circle,

$$2\int_0^a \left\{ \int_0^{\sqrt{a^2-x^2}} \left\{ \ln\left(c^2+4a^2\right) - \ln\left(c^2+a^2\right) \right\} dy \right\} dx = \frac{\pi}{2}\,a^2 \ln\left(\frac{c^2+4a^2}{c^2+a^2}\right).$$

The following two integrals are very difficult, if one only sticks to rectangular coordinates. But even in polar coordinates each of these two integrals are very difficult to compute, though nothing in comparison with the rectangular variant.

We shall of course start by a geometric analysis, because the integrand is the same in both cases. We can therefore join the two integrations over one single one over the domain $B_1 \cup B_2$, which again is more suitable for a polar description:

$$\left\{ (\varrho, \varphi) \;\middle|\; \varrho \in [a, 2a],\, \varphi \in \left[0, \frac{\pi}{2}\right] \right\}.$$

By using this trick we get by insertion,

$$\begin{aligned}
&\int_A \frac{z}{c^2+x^2+y^2+z^2}\,d\Omega \\
&\quad = \frac{\pi}{2}\,a^2 \ln\left(\frac{c^2+4a^2}{c^2+a^2}\right) + 2\int_0^{\frac{\pi}{2}} \left(\int_a^{2a} \left\{ \ln\left(c^2+4a^2\right) - \ln\left(c^2+\varrho^2\right) \right\} \varrho\,d\varrho \right) d\varphi,
\end{aligned}$$

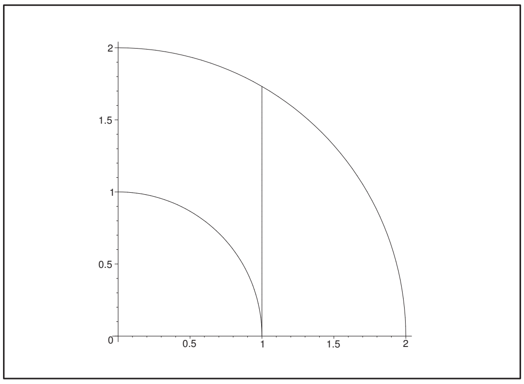

Figure 33: The domains B_1 and B_2 in the meridian half plane.

and the following computations are reduced to variants of those from the second and the third variant.

REMARK. To my knowledge the full computation in rectangular coordinates without any trick has only been carried through once. We also tried to use MAPLE in an earlier version, at that did not work at all. The reason is that one has to apply a dirty rectangular trick at some place, which cannot be foreseen by the computer. $\Diamond$

Example 3.5 *Let a be a positive constant, and let*

$$A = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; x^2+y^2+z^2 \le a^2,\; z \le \sqrt{\frac{x^2+y^2}{3}} \right\}.$$

1) Sketch a meridian half plane, and explain why A is given in spherical coordinates (r,θ,φ) by

$$r \in [0,a], \qquad \theta \in \left[\frac{\pi}{3},\pi\right], \qquad \varphi \in [0,2\pi].$$

2) Compute the space integral

$$\int_A (x^2+z^2)\,d\Omega.$$

A Space integral in spherical coordinate.

D The space integral is here calculated in four variants.

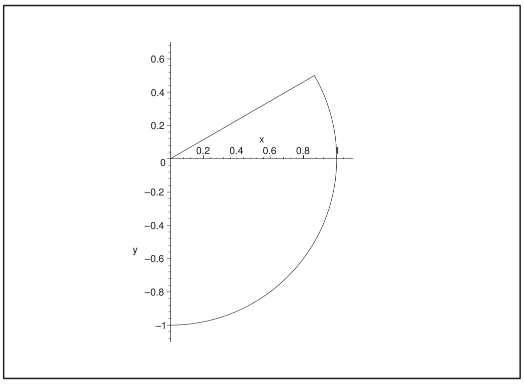

Figure 34: The meridian cut $A^\star$ for $a=1$.

I 1) In the meridian half plane the cut $A^\star$ has the line $z = \dfrac{1}{\sqrt{3}}\,\varrho$ as an upper bound, corresponding to $\theta \in \left[\dfrac{\pi}{3},\pi\right]$. The other variables are not restricted further, so A is given in spherical coordinates by

$$r \in [0,a], \qquad \theta \in \left[\frac{\pi}{3},\pi\right], \qquad \varphi \in [0,2\pi].$$

2) The space integral is here computed in four variants.

First variant. *Direct insertion:*

$$\int_A (x^2+z^2)\,d\Omega = \int_0^{2\pi}\left\{\int_{\frac{\pi}{3}}^{\pi}\left(\int_0^a \left(r^2\sin^2\theta\cos^2\varphi + r^2\cos^2\theta\right) r^2\sin\theta\,dr\right)d\theta\right\}d\varphi$$

$$= \frac{a^5}{5}\int_0^{2\pi}\left\{\int_{\frac{\pi}{3}}^{\pi}\left\{(1-\cos^2\theta)\cos^2\varphi+\cos^2\theta\right\}\sin\theta\right\}d\varphi$$

$$= \frac{a^5}{5}\int_0^{2\pi}\left\{\left[-\cos\theta+\frac{1}{3}\cos^3\theta\right]_{\frac{\pi}{3}}^{\pi}\cos^2\varphi+\left[-\frac{1}{3}\cos^3\theta\right]_{\frac{\pi}{3}}^{\pi}\right\}d\varphi$$

$$= \frac{a^5}{5}\int_0^{2\pi}\left\{\left(1-\frac{1}{3}+\frac{1}{2}-\frac{1}{24}\right)\cos^2\varphi+\left(\frac{1}{3}+\frac{1}{24}\right)\right\}d\varphi$$

$$= \frac{a^5}{5}\left\{\left(\frac{3}{2}-\frac{9}{24}\right)\pi+\frac{9}{24}\cdot 2\pi\right\} = \frac{a^5}{5}\,\pi\left\{\frac{3}{2}-\frac{3}{8}+\frac{3}{4}\right\}$$

$$= \frac{3\pi}{5}\,a^5\left\{\frac{1}{2}+\frac{1}{8}\right\} = \frac{3\pi}{5}\cdot a^5\cdot\frac{5}{8} = \frac{3\pi}{8}\,a^5.$$

Second variant. *A small reduction:*

It follows from $x^2+z^2=r^2-y^2$ that

$$\int_A (x^2+z^2)\,d\Omega = \int_A (r^2-y^2)d\Omega$$

$$= \int_0^{2\pi}\left\{\int_{\frac{\pi}{3}}^{\pi}\left(\int_0^a r^2\left(1-\sin^2\theta\sin^2\varphi\right)r^2\sin\theta\,dr\right)d\theta\right\}d\varphi$$

$$= \frac{a^5}{5}\int_{\frac{\pi}{3}}^{\pi}\left\{2\pi\sin\theta-\pi\sin^3\theta\right\}\,d\theta$$

$$= \pi\cdot\frac{a^5}{5}\left\{[-2\cos\theta]_{\frac{\pi}{3}}^{\pi}-\int_{\frac{\pi}{3}}^{\pi}\left(1-\cos^2\theta\right)\,\sin\theta\,d\theta\right\}$$

$$= \pi\cdot\frac{a^5}{5}\left\{2+1+\left[\cos\theta-\frac{1}{3}\cos^3\theta\right]_{\frac{\pi}{3}}^{\pi}\right\} = \pi\cdot\frac{a^5}{5}\cdot\left\{3+\left(-1+\frac{1}{3}-\frac{1}{2}+\frac{1}{24}\right)\right\}$$

$$= \pi\cdot\frac{a^5}{5}\left(\frac{3}{2}+\frac{3}{8}\right) = \pi\cdot\frac{a^5}{5}\cdot\frac{15}{8} = \frac{3\pi}{8}\,a^5.$$

Third variant. *A symmetric argument:*

For symmetric reasons,

$$\int_A (x^2+z^2)\,d\Omega = \int_A (y^2+z^2)\,d\Omega = \frac{1}{2}\int_A\left\{(x^2+y^2+z^2)+z^2\right\}\,d\Omega$$

$$= \frac{1}{2}\int_0^{2\pi}\left\{\int_{\frac{\pi}{3}}^{\pi}\left(\int_0^a \left(r^2+r^2\cos^2\theta\right)\,r^2\sin\theta\,dr\right)d\theta\right\}d\varphi$$

$$= \frac{1}{2}\cdot 2\pi\int_{\frac{\pi}{3}}^{\pi}(1+\cos^2\theta)\sin\theta\,d\theta\cdot\int_0^a r^4\,dr = \pi\left[-\cos\theta-\frac{1}{3}\cos^3\theta\right]_{\frac{\pi}{3}}^{\pi}\cdot\frac{a^5}{5}$$

$$= \pi\cdot\frac{a^5}{5}\left\{1+\frac{1}{3}+\frac{1}{2}+\frac{1}{3\cdot 8}\right\} = \pi\cdot\frac{a^5}{5}\left(\frac{3}{2}+\frac{3}{8}\right) = \pi\cdot\frac{a^5}{5}\cdot\frac{15}{8} = \frac{3\pi}{8}\,a^5.$$

Fourth variant. *The slicing method.*

At the height $z\in\,]-a,0]$ the body A is cut into a disc D_z given by

$$0\le\varrho\le\sqrt{a^2-z^2}.$$

If instead $z \in \left]0, \frac{a}{2}\right[$, then A is cut into an annulus D_z given by

$$\sqrt{3}\,z \le \varrho \le \sqrt{a^2-z^2}.$$

For symmetric reasons we have for any $z \in \left]-a, \frac{a}{2}\right[$ that

$$\int_{D_z} (x^2+z^2)\,dS = \int_{D_z} (y^2+z^2)\,dS = \int_{D_z} \left\{\frac{1}{2}(x^2+y^2)+z^2\right\} dS.$$

Then we get

$$\begin{aligned}
&\int_A (x^2+z^2)\,d\Omega \\
&= \int_{-a}^{0}\left\{\int_{D_z}\left\{\frac{1}{2}(x^2+y^2)+z^2\right\}dS\right\}dz \int_0^{\frac{a}{2}}\left\{\int_{D_z}\left\{\frac{1}{2}(x^2+y^2)+z^2\right\}dS\right\}dz \\
&= \int_{-a}^{0}\left\{\int_0^{2\pi}\left(\int_0^{\sqrt{a^2-z^2}}\frac{1}{2}\,\varrho^2\cdot\varrho\,d\varrho\right)d\varphi + z^2\,\mathrm{areal}(D_z)\right\}dz \\
&\qquad + \int_0^{\frac{a}{2}}\left\{\int_0^{2\pi}\left(\int_{\sqrt{3}\,z}^{\sqrt{a^2-z^2}}\frac{1}{2}\,\varrho^2\cdot\varrho\,d\varrho\right)d\varphi + z^2\,\mathrm{areal}(D_z)\right\}dz,
\end{aligned}$$

i.e.

$$\begin{aligned}
&\int_A (x^2+z^2)\,d\Omega \\
&= \int_{-a}^{0}\left\{2\pi\left[\frac{1}{8}\,\varrho^4\right]_0^{\sqrt{a^2-z^2}} + z^2\pi\left(a^2-z^2\right)\right\}dz \\
&\qquad + \int_0^{\frac{a}{2}}\left\{2\pi\left[\frac{1}{8}\,\varrho^4\right]_{\sqrt{3}\,z}^{\sqrt{a^2-z^2}} + z^2\pi\left\{(a^2-z^2)-3z^2\right\}\right\}dz \\
&= \frac{\pi}{4}\int_{-a}^{0}\left\{(a^2-z^2)^2+4z^2(a^2-z^2)\right\} \\
&\qquad + \frac{\pi}{4}\int_0^{\frac{a}{2}}\left\{(a^2-z^2)^2-9z^4+4z^2(a^2-4z^2)\right\}dz \\
&= \frac{\pi}{2}\int_{-a}^{0}\left\{a^4-2a^2z^2+z^4+4a^2z^2-4z^4\right\}dz \\
&\qquad + \frac{\pi}{4}\int_0^{\frac{a}{2}}\left\{a^4-2a^2z^2+z^4-9z^4+4a^2z^2-16z^4\right\}dz \\
&= \frac{\pi}{4}\int_{-a}^{0}\left\{a^4+2a^2z^2-3z^4\right\}dz + \frac{\pi}{4}\int_0^{\frac{a}{2}}\left\{a^4+2a^2z^2-24z^4\right\}dz \\
&= \frac{\pi}{4}\left\{\left[a^4z+\frac{2}{3}\,a^2z^3-\frac{3}{5}\,z^5\right]_{-a}^{0} + \left[a^4z+\frac{2}{3}\,a^2z^3-\frac{24}{5}\,z^5\right]_0^{\frac{a}{2}}\right\} \\
&= \frac{\pi}{4}\left\{a^5+\frac{2}{3}\,a^5-\frac{3}{5}\,a^5+\frac{1}{2}\,a^5-\frac{24}{5\cdot 32}\,a^5\right\} \\
&= \frac{\pi}{4}\,a^5\left\{1+\frac{2}{3}-\frac{3}{5}+\frac{1}{2}+\frac{1}{12}-\frac{3}{20}\right\} = \frac{\pi}{4}\,a^5\left\{\frac{3}{2}+\frac{9}{12}-\frac{3}{4}\right\} = \frac{3\pi}{8}\,a^5.
\end{aligned}$$

4 Volume

Example 4.1 *Set up a formula for the volume of the ellipsoid by applying that an ellipse of half axes a and b has the area πab.*

A Volume of an ellipsoid found by a space integral.

D Use the slicing method and describe the ellipse for every z, and continue by computing the corresponding space integral.

I Let the ellipsoid be given by the inequality

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1.$$

For any fixed $z\in[-c,c]$ let $B(z)$ denote the ellipse (in (x,y)-coordinates) given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1-\frac{z^2}{c^2}.$$

When $z \in]-c,c[$, this describes an ellipse with the half axes

$$a\sqrt{1-\frac{z^2}{c^2}} \quad \text{og} \quad b\sqrt{1-\frac{z^2}{c^2}}.$$

Then by the slicing method,

$$\begin{aligned}
\text{vol}(B) &= \int_{-c}^{c}\left\{\int_{B(z)} dx\,dy\right\} dz = \int_{-c}^{c} \text{area}(B(z))\,dz \\
&= \int_{-c}^{c} \pi a\sqrt{1-\frac{z^2}{c^2}}\cdot b\sqrt{1-\frac{z^2}{c^2}}\,dz = \pi ab\int_{-c}^{c}\left(1-\frac{z^2}{c^2}\right)dz \\
&= 2\pi abc\int_0^1 (1-t^2)\,dt = 2\pi abc\left[t-\frac{1}{3}\,t^3\right]_0^1 = \frac{4\pi}{3}\,abc.
\end{aligned}$$

C As a weak control we know that for the solid ball of radius $r=a=b=c$ we get the well-known volume $\dfrac{4\pi}{3}\,r^3$.

Example 4.2 *Let B be a closed domain in the (x,y)-plane, and let P_0 be a point of z-coordinate h. Draw linear segments through P_0 and the points of B. The union of these are making up a (solid) cone. One calls B the base of the cone and h is the height of the cone.*

1) A plane of constant $z \in [0,h]$ intersects the cone in a plane domain $B(z)$. Show that the area of $B(z)$ is equal to the area of B multiplied by the factor $\left(1-\dfrac{z}{h}\right)^2$. (Consider e.g. elements of area which correspond to each other by the segments mentioned above).

2) Prove that the volume of the cone is $\dfrac{1}{3}\,hA$, where A is the area of the base B.

3) Prove that the z-coordinate of the centre of gravity is given by $\dfrac{1}{4}\,h$.

A The volume of a cone found by a space integral.

D Follow the guidelines given above.

I 1) By considering a rectangular element of area in B we see by using similar triangles that every length in the corresponding element of area in $B(z)$ is diminished by the factor

$$\frac{h-z}{h} = 1-\frac{z}{h}.$$

The element of area is determined by two lengths (“length” and “breadth”), so the area is reduced by the factor $\left(1-\dfrac{z}{h}\right)^2$, i.e.

$$\text{area}(B(z)) = \left(1-\frac{z}{h}\right)^2 \text{area } B = \left(1-\frac{z}{h}\right)^2 A.$$

2) Using the result from 1) we get by the slicing method,

$$\begin{aligned}\operatorname{vol}(K) &= \int_K d\Omega = \int_0^h \left\{ \int_{B(z)} dx\,dy \right\} dz = \int_0^h \text{ area}((B(z))\,dz \\ &= \int_0^h \left(1-\frac{z}{h}\right)^2 A\,dz = Ah\left[-\frac{1}{3}\left(1-\frac{z}{h}\right)^3\right]_0^h = \frac{1}{3}\,hA.\end{aligned}$$

3) Let the cone be homogeneously coated (density $\mu > 0$). Then the mass is

$$M = \mu \operatorname{vol}(K) = \frac{1}{3}\,\mu h A.$$

The z-coordinate ζ of the centre of gravity is given by

$$M \cdot \zeta = \mu \int_K z\,d\Omega,$$

thus

$$\begin{aligned}\zeta &= \frac{\mu}{M}\int_K z\,d\Omega = \frac{\mu}{\frac{1}{3}\,\mu h A}\int_0^h z\cdot \text{areal}(B(z))\,dz = \frac{3}{hA}\int_0^h z\left(1-\frac{z}{h}\right)^2 A\,dz \\ &= 3\int_0^h \frac{z}{h}\left(1-\frac{z}{h}\right)^2 dz = 3h\int_0^1 (1-t)t^2\,dt = 3h\int_0^1 (t^2-t^3)\,dt \\ &= 3h\left[\frac{1}{3}\,t^3 - \frac{1}{4}\,t^4\right]_0^1 = 3h\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{h}{4}.\end{aligned}$$

Example 4.3 *Find the volume of the point set*

$$\Omega = \{(x,y,z) \mid x^2+y^2 \le a^2,\ |x| \le a+y,\ 0 \le z \le x^2+y^2\}.$$

Then compute the space integral

$$\int_\Omega (xy+1)\,d\Omega.$$

A Volume and space integral.

D Sketch the projection B of Ω onto the (x,y)-plane. Find the volume and the space integral.

I The volume is

$$\begin{aligned}\operatorname{vol}(\Omega) &= \int_B (x^2+y^2)\,dx\,dy = \int_0^\pi \left\{\int_0^a \varrho^2\cdot\varrho\,d\right\} d\varphi + \int_{-a}^0 \left\{\int_{-a-y}^{a+y} (x^2+y^2)\,dx\right\} dy \\ &= \pi\cdot\frac{a^4}{4} + \int_{-a}^0 \left[\frac{1}{3}\,x^3 + y^2 x\right]_{x=-a-y}^{a+y} dy = \frac{\pi a^4}{4} + \int_{-a}^0 \left\{\frac{2}{3}\,(a+y)^3 + 2y^2(a+y)\right\} dy \\ &= \frac{\pi a^4}{4} + \left[\frac{1}{6}\,(a+y)^4 + \frac{2}{3}\,ay^3 + \frac{1}{2}\,y^4\right]_{-a}^0 = \frac{\pi a^4}{4} + \frac{1}{6}\,a^4 + \frac{2}{3}\,a^4 - \frac{1}{2}\,a^4 \\ &= a^4\left(\frac{\pi}{4}+\frac{1}{6}+\frac{2}{3}-\frac{1}{2}\right) = a^4\left(\frac{\pi}{4}+\frac{1}{3}\right).\end{aligned}$$

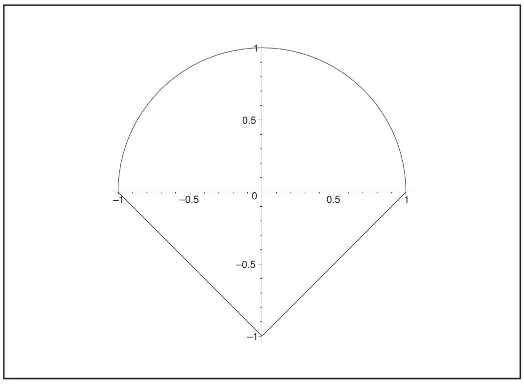

Figure 35: The projection B of Ω for $a = 1$.

Due to the symmetry with respect to the Y-axis, we get for the space integral that

$$\int_{\Omega} (xy+1)\, d\Omega = \int_{B} xy(x^2+y^2)\, dx\, dy + \mathrm{vol}(\Omega) = 0 + \mathrm{vol}(\Omega) = a^4 \left(\frac{\pi}{4} + \frac{1}{3} \right).$$

Example 4.4 *Let $\mathcal{C}$ denote the cylindric surface which generators are parallel to the Z-axis and the intersection curve of which with the (x,y)-plane has the equation $y^2 = x$. Find the volume of the point set Ω, which is bounded by*

1) the cylindric surface $\mathcal{C}$,

2) the (x,y)plane, and

3) the plane of the equation $2x + 2y + z = 4$.

A Volume.

D Sketch Ω, or at least the projection D of Ω onto the (x,y)-plane.

I It follows from

$$D = \{(x,y) \mid -2 \le y \le 1,\; y^2 \le x \le 2 - y\},$$

and

$$0 \le z \le 4 - 2x - 2y = 2(2 - x - y),$$

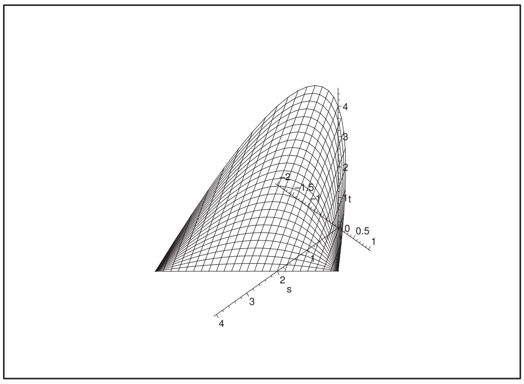

Figure 36: The body Ω.

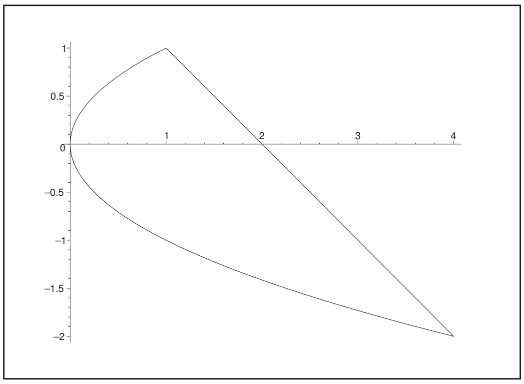

Figure 37: The projection D of Ω onto the (x, y)-plane.

that

$$\begin{aligned}
\text{vol}(\Omega) &= \int_D 2(2-x-y)\,dx\,dy = \int_{-2}^{1}\left\{\int_{y^2}^{2-y} 2(2-x-y)dx\right\}dy \\
&= \int_{-2}^{1}\left[-(2-x-y)^2\right]_{x=y^2}^{2-y} dy = \int_{-2}^{1}\left(2-y-y^2\right)^2\,dy \\
&= \int_{-2}^{1}(y+2)^2(y-1)^2\,dy = \int_{-\frac{3}{2}}^{\frac{3}{2}}\left(t+\frac{3}{2}\right)^2\left(t-\frac{3}{2}\right)^2 dt = 2\int_0^{\frac{3}{2}}\left(t^2-\frac{9}{4}\right)^2 dt \\
&= 2\int_0^{\frac{3}{2}}\left(t^4-\frac{9}{2}t^2+\frac{81}{16}\right)dt = 2\left\{\frac{1}{5}\left(\frac{3}{2}\right)^5-\frac{3}{2}\left(\frac{3}{2}\right)^3+\frac{81}{16}\cdot\frac{3}{2}\right\} \\
&= \frac{2}{32}\left\{\frac{1}{5}\cdot 3^5-2\cdot 3^4+3^5\right\} = \frac{3^4}{16}\cdot\left\{\frac{3}{5}+1\right\} = \frac{81}{16}\cdot\frac{8}{5} = \frac{81}{10}.
\end{aligned}$$

Example 4.5 *Let*

$$f(x,y) = \ln\left(2-2x^2-3y^2\right)+2-4x^2-6y^2,$$

and let B *be that part of the* (x,y)*-plane, in which* $f(x,y) \geq 0$. *Let* L *denote the point set in the space which is given by*

$$(x,y) \in B, \qquad 0 < leqz \leq f(x,y).$$

Find the volume of L *by the slicing method.*

A Volume.

D Consider $f(x,y) = \ln(2-2x^2-3y^2)+2-4x^2-6y^2$ as a function in one single variable.

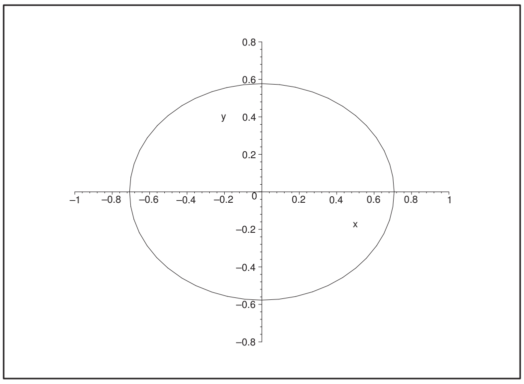

Figure 38: The domain B, in which $f(x,y) \geq 0$.

I Since $f(x,y)=0$, for $2x^2+3y^2=1$, the domain B is bounded by the ellipse

$$\left(\frac{x}{\frac{1}{\sqrt{2}}}\right)^2+\left(\frac{y}{\frac{1}{\sqrt{3}}}\right)^2=1.$$

This ellipse has the half axes $\dfrac{1}{\sqrt{2}}$ and $\dfrac{1}{\sqrt{3}}$ and the area $\dfrac{\pi}{\sqrt{6}}$.

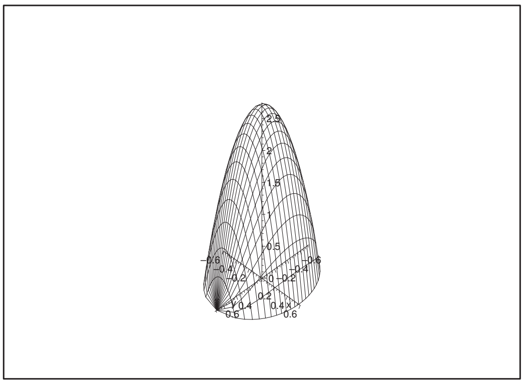

Figure 39: The body L.

The function $f(x,y)$ is in reality only a function in $t=1-2x^2-3y^2$, $t\in[0,1]$, since we have by this substitution

$$z=f(x,y)=F(t)=\ln(1+t)+2t, \qquad t\in[0,1].$$

When $t\in[0,1]$ is fixed, then $2x^2+3y^2\leq 1-t$ describes an elliptic disc A_t of area

$$\text{area}\,(A_t)=\frac{\pi}{\sqrt{6}}\,(1-t),$$

thus we get the volume by the slicing method,

$$\begin{aligned}\mathrm{vol}(L) &= \int_0^1 \mathrm{area}\,(A_t)\cdot\frac{dz}{dt}\,dt = \int_0^1 \frac{\pi}{\sqrt{6}}\,(1-t)\cdot\left\{\frac{1}{1+t}+2\right\}dt\\ &= \frac{\pi}{\sqrt{6}}\int_0^1\left\{\frac{2-(1+t)}{1+t}+2-2t\right\}dt = \frac{\pi}{\sqrt{6}}\left[\ln(1+t)-t+2t-t^2\right]_0^1 = \frac{\pi}{\sqrt{6}}\,\ln 2.\end{aligned}$$

Example 4.6 *Find the volume of the point set*

$$Q = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2\le 2,\ 0\le z\le 4-\left(x^2+y^4\right)^2\}.$$

A Volume.

D Sketch the set and just compute.

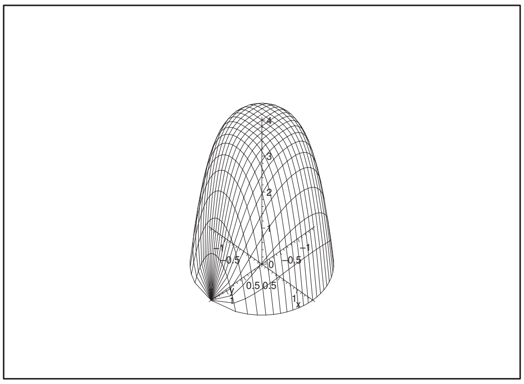

Figure 40: The point set Q.

I The set Q is cut at height $z\in[0,4]$ in a disc of radius $\sqrt[4]{4-z}$. Then by the slicing method,

$$\mathrm{vol}(Q) = \int_0^4 \pi\sqrt{4-z}\,dz = \pi\left[-\frac{2}{3}\,\left(\sqrt{4-z}\right)^3\right]_0^4 = \frac{2}{3}\,\pi\cdot(\sqrt{4})^3 = \frac{16}{3}\,\pi.$$

ALTERNATIVELY we first integrate with respect to z,

$$\begin{aligned}\mathrm{vol}(Q) &= \int_{\overline{K}(\mathbf{0};\sqrt{2})}\left\{4-(x^2+y^2)^2\right\}\,dx\,dy = 4\,\mathrm{area}\left(\overline{K}(\mathbf{0};\sqrt{2})\right)-2\pi\int_0^{\sqrt{2}}\varrho^4\cdot\varrho\,d\varrho\\ &= 4\cdot 2\pi-2\pi\left\{\frac{(\sqrt{2})^6}{6}\right\} = 8\pi-\frac{8\pi}{3} = \frac{16\pi}{3}.\end{aligned}$$

Example 4.7 *Let $B(a)$ denote the bounded point set in the plane which is bounded by the parabola $y = x^2$ and the line $y = a$. Let B denote the unbounded point set which is defined by the inequalities $y \geq x^2$. Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be given by*

$$f(x,y) = |x| \exp\left(x^2 - 2y\right),$$

and put

$$A(a) = \{(x,y,z) \mid (x,y) \in B(a),\, 0 \leq z \leq f(x,y)\}.$$

1) Find the volume of $A(a)$.

2) Prove that the improper plane integral

$$\int_B f(x,y)\,dS$$

is convergent, and find its value.

A Volume and improper plane integral.

D Sketch $B(a)$ and B; find vol $A(a)$, and compute the improper plane integral.

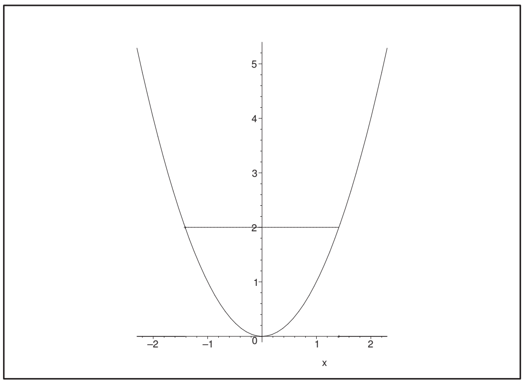

Figure 41: The parabola with the truncation at $y = a = 2$.

I 1) We get by direct computation,

$$\begin{aligned}
\mathrm{vol}(A(a)) &= \int_{B(a)} f(x,y)\,dS = 2\int_0^a \left\{\int_0^{\sqrt{y}} x \cdot e^{x^2} e^{-2y}\,dx\right\} dy \\
&= 2\int_0^a e^{-2y} \left[\frac{1}{2}\,e^{x^2}\right]_{x=0}^{\sqrt{y}} dy = \int_0^a e^{-2y}\left(e^y - 1\right) dy \\
&= \int_0^a \left(e^{-y} - e^{-2y}\right) dy = \left[-e^{-y} + \frac{1}{2}\,e^{-2y}\right]_0^a = \frac{1}{2} - e^{-a} + \frac{1}{2}\,e^{-2a}.
\end{aligned}$$

2) Since $f(x,y) \geq 0$, we get

$$\int_B f(x,y)\,dS = \lim_{a\to+\infty} int_{B(a)} f(x,y)\,dS = \lim_{a\to+\infty}\left\{\frac{1}{2} - e^{-a} + \frac{1}{2}\,e^{-2a}\right\} = \frac{1}{2}.$$

5 Moment of inertia and centre of gravity

Example 5.1 *Given the solid ellipsoid*

$$\Omega = \left\{ (x,y,z) \;\middle|\; \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \le 1 \right\}.$$

1) Compute the space integral

$$\int_\Omega \sqrt{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2}\, d\Omega.$$

2) Let Ω be homogeneously coated by a mass, where M denotes the total mass. Find the moment of inertia I_x of Ω with respect to the X-axis expressed by a, b, c and M.

A Space integral; moment of inertia.

D Follow the guidelines.

I 1) By putting

$$(x, y, z) = (au, bv, cw), \qquad u^2 + v^2 + w^2 \leq 1,$$

and then applying spherical coordinates in the (u, v, w)-space we get

$$\begin{aligned}\int_\Omega \sqrt{\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2}\, d\Omega &= abc \int_{\overline{K}(\mathbf{0};1)} \sqrt{u^2+v^2+w^2}\, du\, dv\, dw \\ &= abc \int_0^{2\pi} \left\{ \int_0^\pi \left\{ \int_0^1 \varrho \cdot \varrho^2 \sin\theta\, d\varrho \right\} d\theta \right\} d\varphi = abc \cdot 2\pi \cdot 2 \cdot \frac{1}{4} = abc\pi.\end{aligned}$$

2) It is well-known that the volume is $\operatorname{vol}(\Omega) = \dfrac{4\pi}{3}\, abc$, hence the mass can be written $M = \dfrac{4\pi}{3}\, abc \cdot \mu$, from which we get the density $\mu = \dfrac{3M}{4\pi abc}$.
Due to the symmetry, the moment of inertia with respect to the X-axis is given by

$$\begin{aligned} I_x &= \mu \int_\Omega (y^2 + z^2)\, d\Omega = \mu \int_\Omega y^2\, d\Omega + \mu \int_\Omega z^2\, d\Omega \\ &= \mu b^2(abc) \int_{\overline{K}(\mathbf{0};1)} v^2 du dv dw + \mu c^2(abc) \int_{\overline{K}(\mathbf{0};1)} w^2 du dv dw \\ &= \mu abc(b^2+c^2) \int_{\overline{K}(\mathbf{0};1)} u^2 du dv dw = \mu(b^2+c^2)abc \int_{-1}^1 u^2\pi(1-u^2) du \\ &= 2\mu\pi abc(b^2+c^2) \int_0^1 (u^2-u^4) du = 2 \cdot \frac{3M}{4\pi abc}\, abc(b^2+c^2) \left(\frac{1}{3}-\frac{1}{5}\right) \\ &= \frac{3}{2}\, M\, (b^2+c^2) \cdot \frac{2}{15} = \frac{1}{5}\, M(b^2+c^2). \end{aligned}$$

Example 5.2 *Find the centre of gravity for the part of the intersection of the ball of centrum* $(0, 0, 0)$ *and of radius* $a > 0$ *in the first octant, i.e. given by the inequalities*

$$x \geq 0, \quad y \geq 0, \quad z \geq 0, \quad x^2 + y^2 + z^2 \leq a^2.$$

A Centre of gravity.

D First reduce to the case $a = 1$. Find $\operatorname{vol}(\Omega)$. Compute

$$\xi = \frac{1}{\operatorname{vol}(\Omega)} \int_\Omega x\, d\Omega.$$

It follows by the symmetry that $\xi = \eta = \zeta$.

I Of geometrical reasons we may assume that $a = 1$, thus

$$\Omega = \{(x, y, z) \mid x \geq 0,\, y \geq 0,\, z \geq 0,\, x^2 + y^2 + z^2 \leq 1\}.$$

If (ξ, η, ζ) denotes the centre of gravity for Ω, then $(a\xi, a\eta, a\zeta)$ is the centre of gravity for the initial set of radius a.

It follows clearly by the symmetry that $\xi = \eta = \zeta$.

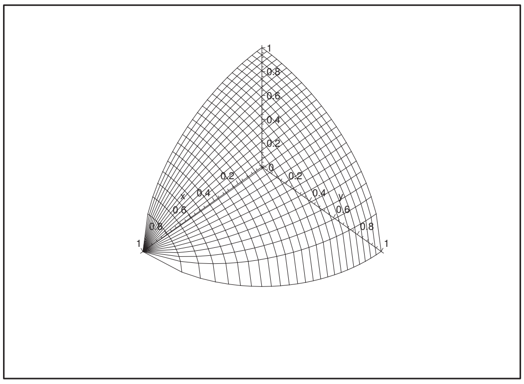

Figure 42: The set Ω for $a = 1$.

Finally,

$$\text{vol}(\Omega) = \frac{1}{8} \cdot \frac{4\pi}{3} \cdot 1^3 = \frac{\pi}{6}.$$

It follows that

$$\begin{aligned} \xi &= \frac{1}{\text{vol}(\Omega)} = \frac{6}{\pi} \int_0^1 x \left\{ \int_{y^2+z^2 \le 1-x^2} dy\, dz \right\} dx \\ &= \frac{6}{\pi} \int_0^1 x \cdot \frac{1}{4}\, \pi \left(1-x^2\right) dx = \frac{3}{2} \int_0^1 \left(x - x^3\right) dx = \frac{3}{2} \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{3}{8}. \end{aligned}$$

Therefore, if $a = 1$, then

$$(\xi, \eta, \zeta) = \left(\frac{3}{8}, \frac{3}{8}, \frac{3}{8} \right).$$

We get for a general $a > 0$,

$$(\xi, \eta, \zeta) = \left(\frac{3}{8}\, a\,,\, \frac{3}{8}\, a\,,\, \frac{3}{8}\, a \right).$$

Example 5.3 *Let R denote a positive constant. Consider the point set*

$$T = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \le z,\, x^2+y^2+z^2 \le R^2,\, x^2+y^2 \le 3z^2\}.$$

1) Explain why T is given in spherical coordinates by

$$r \in [0,R], \qquad \theta \in \left[0, \frac{\pi}{3}\right], \qquad \varphi \in [0,2\pi].$$

2) Compute the space integrals $\int_T 1\,dx\,dy\,dz$ and $\int_T z\,dx\,dy\,dz$.

3) Find the coordinates of the centre of gravity of T.

4) Find the area of the boundary surface of T.

A Spherical coordinates, space integrals, centre of gravity and surface area.

D First make a sketch in the meridian half plane.

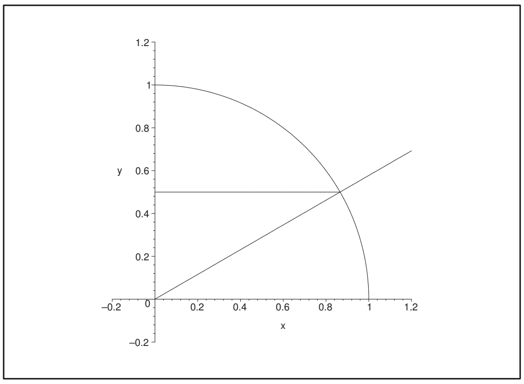

Figure 43: The sketch in the meridian half plane for $R = 1$.

I 1) The sketch of the meridian half plane shows that

$$z \ge 0, \quad r^2 \le R^2, \quad \varrho^2 \le 3z^2, \quad r^2 = \varrho^2 + z^2,$$

in spherical coordinates is expressed by

$$r \in [0,R], \qquad \theta \in \left[0, \frac{\pi}{3}\right], \qquad \varphi \in [0,2\pi].$$

2) The volume is

$$\begin{aligned}
\mathrm{vol}(T) &= \int_T 1\,dx\,dy\,dz = \int_0^{\frac{R}{2}} \pi \cdot 3z^2\,dz + \int_{\frac{R}{2}}^R \pi\left(R^2 - z^2\right)\,dz \\
&= \pi\left(\frac{R}{2}\right)^3 + \pi\left[R^2 z - \frac{1}{3}\,z^3\right]_{\frac{R}{2}}^R = \frac{\pi}{8}\,R^3 + \pi\left\{R^3 - \frac{R^3}{3} - \frac{R^3}{2} + \frac{1}{24}\,R^3\right\} \\
&= \pi R^3\left\{\frac{1}{8} + 1 - \frac{1}{3} - \frac{1}{2} + \frac{1}{24}\right\} = \frac{\pi R^3}{24}\,\{3 + 24 - 8 - 12 + 1\} = \frac{\pi R^3}{3}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_T z\,dx\,dy\,dz &= \int_0^{\frac{R}{2}} z\cdot\pi\cdot 3z^2\,dz + \int_{\frac{R}{2}}^{R} z\cdot\pi\left(R^2-z^2\right)\,dz \\
&= \frac{3\pi}{4}\cdot\left(\frac{R}{2}\right)^4 + \pi\left[\frac{R^2}{2}\,z^2 - \frac{1}{4}\,z^4\right]_{\frac{R}{2}}^{R} \\
&= \frac{3\pi}{64}\,R^4 + \pi\left\{\frac{R^4}{2} - \frac{R^4}{4} - \frac{R^4}{8} + \frac{R^4}{64}\right\} \\
&= \frac{\pi R^4}{64}\{3+32-16-8+1\} = \frac{12\pi R^4}{64} = \frac{3\pi}{16}\,R^4.
\end{aligned}$$

3) Of symmetric reasons the centre of gravity must lie on the Z-axis, so $\xi = \eta = 0$, and

$$\zeta = \frac{1}{\mathrm{vol}(T)}\int_T z\,dx\,dy\,dz = \frac{3}{\pi R^3}\cdot\frac{3\pi}{16}\,R^4 = \frac{9}{16}\,R,$$

where we have used the results of 2).

4) The boundary curve $\mathcal{M}$ in the meridian half plane is now split up into

$$\mathcal{M}_1: \quad \varrho = \sqrt{3}\cdot z, \quad ds = \sqrt{1+3}\,dz = 2\,dz, \quad z \in \left[0, \frac{R}{2}\right],$$

$$\mathcal{M}_2: \quad \varrho = \sqrt{R^2-z^2}, \quad ds = \frac{R}{\sqrt{R^2-z^2}}\,dz, \qquad z \in \left[\frac{R}{2}, R\right],$$

so the surface area becomes

$$\begin{aligned} 2\pi \int_{\mathcal{M}} P\,ds &= 2\pi \int_0^{\frac{R}{2}} \sqrt{3}\cdot z\cdot 2\,dz + 2\pi \int_{\frac{R}{2}}^{R} \sqrt{R^2-z^2}\cdot \frac{R}{\sqrt{R^2-z^2}}\,dz \\ &= 2\pi\sqrt{3}\,\left[z^2\right]_0^{\frac{R}{2}} + 2\pi R\cdot\frac{R}{2} = 2\sqrt{3}\,\pi\cdot\frac{R^2}{4} + \pi R^2 = \left(1+\frac{\sqrt{3}}{2}\right)\pi R^2. \end{aligned}$$

Example 5.4 *Let Ω denote that part of the closed ball $\overline{K}(\mathbf{0};a)$, which lies above the (x,y)-plane and inside a cylindric surface with its generators parallel to the Z-axes through the curve in the (x,y)-plane given by the equation*

$$\varrho = a\sqrt{\cos(2\varphi)}, \qquad -\frac{\pi}{4} \le \varphi \le \frac{\pi}{4}.$$

1) Find the volume of $\overline{\Omega}$.

2) Find the z-coordinate of the centre of gravity for $\overline{\Omega}$.

A Volume; centre of gravity.

D Sketch Ω and compute $\mathrm{vol}(\Omega)$. Find the centre of gravity.

I 1) Since $z = +\sqrt{a^2-\varrho^2}$ on the shell, we get

$$\begin{aligned} \mathrm{vol}(\Omega) &= \int_B \sqrt{a^2-\varrho^2}\,dS = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left\{\int_0^{a\sqrt{\cos 2\varphi}} \sqrt{a^2-\varrho^2}\cdot\varrho\,d\varrho\right\} d\varphi \\ &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[-\frac{1}{2}\cdot\frac{1}{3}\,\left(a^2-\varrho^2\right)^{\frac{3}{2}}\right]_{\varrho=0}^{a\sqrt{\cos 2\varphi}} d\varphi = \frac{1}{3}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left\{a^3 - a^3(1-\cos 2\varphi)^{\frac{3}{2}}\right\} d\varphi \\ &= 2\cdot\frac{a^3}{3}\int_0^{\frac{\pi}{4}} \left\{1-(2\sin^2\varphi)^{\frac{3}{2}}\right\} d\varphi = \frac{2}{3}\,a^3 \int_0^{\frac{\pi}{4}} \left\{1-2\sqrt{2}\sin^3\varphi\right\} d\varphi \\ &= \frac{2}{3}\,a^3\cdot\frac{\pi}{4} - \frac{2}{3}\,a^3\cdot 2\sqrt{2}\int_0^{\frac{\pi}{4}} \left(1-\cos^2\varphi\right)\sin\varphi\,d\varphi \\ &= \frac{\pi}{6}\,a^3 + \frac{4\sqrt{2}}{3}\,a^3\left[\cos\varphi - \frac{1}{3}\,\cos^3\varphi\right]_0^{\frac{\pi}{4}} = \frac{\pi}{6}\,a^3 + \frac{4\sqrt{2}}{3}\,a^3\left(\frac{1}{\sqrt{2}} - \frac{1}{6\sqrt{2}} - \frac{2}{3}\right) \\ &= \frac{\pi}{6}\,a^3 + \frac{4}{3}\cdot\frac{5}{6}\,a^3 - \frac{8\sqrt{2}}{9}\,a^3 = a^3\left(\frac{\pi}{6} + \frac{10}{9} - \frac{8\sqrt{2}}{9}\right). \end{aligned}$$

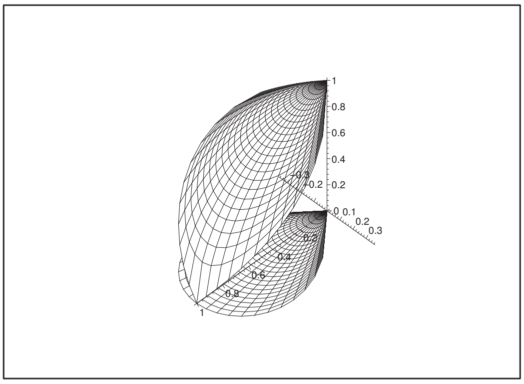

Figure 44: The domain Ω for $a=1$.

2) We have due to the symmetry,

$$\int_\Omega y\,d\Omega = \int_B y\sqrt{a^2-\varrho^2}\,dS = 0.$$

Furthermore,

$$\begin{aligned}
\int_\Omega z\,d\Omega &= \int_B\left\{\int_0^{\sqrt{a^2-\varrho^2}} z\,dz\right\}dS = \frac{1}{2}\int_B\left(a^2-\varrho^2\right)\,dS \\
&= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left\{\int_0^{a\sqrt{\cos 2\varphi}}\left(a^2-\varrho^2\right)\,\varrho\right\}d\varphi = \int_0^{\frac{\pi}{4}}\left[\frac{1}{2}\,a^2\varrho^2-\frac{1}{4}\,\varrho^4\right]_0^{a\sqrt{\cos 2\varphi}}d\varphi \\
&= \frac{a^4}{4}\int_0^{\frac{\pi}{4}}\{2\cos 2\varphi-\cos^2 2\varphi\}d\varphi \\
&= \frac{a^4}{4}\left[\sin 2\varphi\right]_0^{\frac{\pi}{4}}-\frac{a^4}{4}\cdot\frac{1}{2}\int_0^{\frac{\pi}{4}}\{1+\cos 4\varphi\}d\varphi = \frac{a^4}{4}-\frac{a^4\pi}{32} = \frac{a^4}{32}\,(8-\pi).
\end{aligned}$$

Finally, we get by interchanging the order of integration that

$$\begin{aligned}
\int_\Omega x\,d\Omega &= \int_B x\sqrt{a^2-\varrho^2}\,dS = 2\int_0^{\frac{\pi}{4}}\cos\varphi\left\{\int_0^{a\sqrt{\cos 2\varphi}}\sqrt{a^2-\varrho^2}\cdot\varrho^2\,d\varrho\right\}d\varphi \\
&= 2a^4\int_0^{\frac{\pi}{4}}\cos\varphi\left\{\int_0^{\sqrt{\cos 2\varphi}}t^2\sqrt{1-t^2}\,dt\right\}d\varphi \\
&= 2a^4\int_0^1 t^2\sqrt{1-t^2}\left\{\int_0^{\frac{1}{2}\,\mathrm{Arccos}(t^2)}\cos\varphi\,d\varphi\right\}dt \\
&= 2a^4\int_0^1 t^2\sqrt{1-t^2}\,\sin\left(\frac{1}{2}\,\mathrm{Arccos}(t^2)\right)dt \\
&= 2a^4\int_0^1 t^2\sqrt{1-t^2}\cdot\sqrt{1-\cos\left(2\cdot\frac{1}{2}\,\mathrm{Arccos}\,(t^2)\right)}dt \\
&= 2a^4\int_0^1 t^2\sqrt{1-t^2}\cdot\sqrt{1-t^2}\,dt = 2a^4\int_0^1 t^2(1-t^2)\,dt = 2a^4\left\{\frac{1}{3}-\frac{1}{5}\right\} = \frac{4}{15}\,a^4.
\end{aligned}$$

The centre of gravity is

$$\xi = \frac{1}{\mathrm{vol}(\Omega)} \int_{\Omega} \mathbf{x}\, d\Omega = \frac{a}{\frac{\pi}{6}+\frac{10}{9}-\frac{8\sqrt{2}}{9}} \left(\frac{4}{15}, 0, \frac{8-\pi}{32} \right).$$

6 Improper space integrals

Example 6.1 *Check in each of the following cases if the given space integral is convergent or divergent; find the value in case of convergency.*

1) The space integral $\int_A \dfrac{1}{\sqrt{x+y+z}}\,d\Omega$, *where the domain of integration* A *is described by*

$$x \geq 0, \quad y \geq 0, \quad x+y \leq 1, \quad 0 \leq z \leq x+y.$$

2) The space integral $\int_A \dfrac{z^3}{(x^2+y^2+z^2)^{\frac{3}{2}}}\,d\Omega$, *where the domain of integration* A *is described by*

$$x^2+y^2 \leq z^4 \leq 1, \qquad z \geq 0.$$

3) The space integral $\int_A \dfrac{1}{1+xyz}\,d\Omega$, *where the domain of integration* A *is described by*

$$r \in [0,+\infty[, \quad \theta \in \left[\frac{\pi}{3},\frac{\pi}{2}\right], \quad \varphi \in \left[\frac{\pi}{6},\frac{\pi}{3}\right].$$

4) The space integral $\int_A \dfrac{1}{z^2}\,d\Omega$, *where the domain of integration* A *is described by*

$$\varrho \leq z \leq h, \qquad \varphi \in [0,2\pi].$$

5) The space integral $\int_A z\,\exp\left(-\left(x^2+y^2+z^2\right)\right)d\Omega$, *where the domain of integration is* $A = \mathbb{R}^3$.

6) The space integral $\int_A \exp\left(-2x^2-3y^2-6z^2\right)d\Omega$, *where the domain of integration is* $A = \mathbb{R}^3$.

A Improper space integrals. Note that the domain of integration is described in various coordinate systems according to the usual conventions.

D Whenever possible, sketch the projection of the domain of integration onto the XY-plane. Explain why the space integral is improper (i.e. if the integrand is not defined in all points and/or if the domain of integration A is unbounded). Truncate the domain: compute the space integral over the truncated domain and finally take the limit.

I 1) The domain A is bounded. The integrand is not defined for $x+y+z=0$, i.e. at the point $(0,0,0) \in A$. The integrand is positive elsewhere, so we can choose the truncation

$$A_t = \{(x,y,z) \mid x \geq 0,\, y \geq 0,\, t \leq x+y \leq 1,\, 0 \leq z \leq x+y\} \quad t \in]0,1[.$$

Let

$$B_t = \{(x,y) \mid x \geq 0,\, y \geq 0,\, t \leq x+y \leq 1\} \quad \text{og} \quad B = B_0$$

denote the projection of A_t onto the XY-plane. Then

$$\begin{aligned}\int_{A_t} \frac{1}{\sqrt{x+y+z}}\,d\Omega &= \int_{B_t} \left\{\int_0^{x+y} \frac{1}{\sqrt{x+y+z}}\,dz\right\} dx\,dy \\ &= \int_{B_t} \left[2\sqrt{x+y+z}\right]_{z=0}^{x+y} dx\,dy = 2(\sqrt{2}-1)\int_{B_t} \sqrt{x+y}\,dx\,dy,\end{aligned}$$

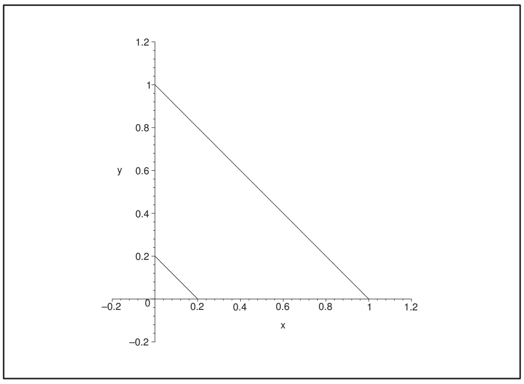

Figure 45: The truncation B_t in the XY-plane of **Example 6.1.1**.

which clearly has a limit for $t \to 0+$.

Hence the improper integral exists, and its value is

$$\begin{aligned}
\int_A \frac{1}{\sqrt{x+y+z}}\, d\Omega &= 2(\sqrt{2}-1)\int_B \sqrt{x+y}\, dx\, dy = 2(\sqrt{2}-1)\int_0^1 \left\{\int_0^{1-x}\sqrt{x+y}\, dy\right\} dx \\
&= 2(\sqrt{2}-1)\int_0^1 \left[\frac{2}{3}\,(x+y)^{\frac{3}{2}}\right]_{y=0}^{1-x} dx = \frac{4}{3}(\sqrt{2}-1)\int_0^1 \left\{1-x^{\frac{3}{2}}\right\} dx \\
&= \frac{4}{3}(\sqrt{2}-1) - \frac{8}{15}(\sqrt{2}-1)\left[x^{\frac{5}{2}}\right]_0^1 = (\sqrt{2}-1)\cdot\frac{12}{15} = \frac{4}{5}(\sqrt{2}-1).
\end{aligned}$$

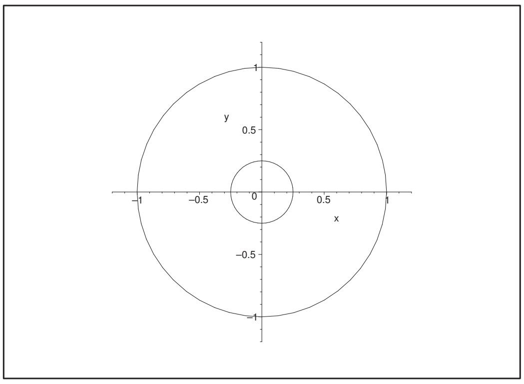

Figure 46: The projection of the truncation B_t in **Example 6.1.2**.

2) The domain is bounded, at the integrand is positive with the exception of $(0,0,0)$, where it is not defined. We choose the truncation

$$A_t = \{(x,y,z) \mid t^t \le x^2+y^2 \le z^4 \le 1\}, \qquad t \in]0,1[,$$

with the projection

$$B_t = \{(x,y) \mid t^2 \le x^2+y^2 \le 1\} = \{(\varrho,\varphi) \mid t \le \varrho \le 1,\, 0 \le \varphi \le 2\pi\}.$$

Then

$$\begin{aligned}
\int_{A_t} \frac{z^3}{(x^2+y^2+z^4)^{\frac{3}{2}}}\,d\Omega &= \int_{B_t}\left\{\int_{\sqrt[4]{x^2+y^2}}^{1} \frac{z^3}{x^2+y^2+z^4)^{\frac{3}{2}}}\,dz\right\}dx\,dy \\
&= \int_{B_t}\left[\frac{1}{4}\cdot(-2)\,\frac{1}{\sqrt{x^2+y^2+z^4}}\right]_{z=\sqrt[4]{x^2+y^2}}^{1} dx\,dy \\
&= \frac{1}{2}\int_{B_t}\left\{\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{x^2+y^2}}-\frac{1}{\sqrt{x^2+y^2+1}}\right\}dx\,dy \\
&= \frac{1}{2}\int_0^{2\pi}\left\{\int_t^1\left\{\frac{1}{\sqrt{2}}\cdot\frac{1}{\varrho}-\frac{1}{\sqrt{\varrho^2+1}}\right\}\varrho\,d\varrho\right\}d\varphi = \pi\int_t^1\left\{\frac{1}{\sqrt{2}}-\frac{\varrho}{\sqrt{\varrho^2+1}}\right\}d\varrho \\
&= \pi\left[\frac{\varrho}{\sqrt{2}}-\sqrt{\varrho^2+1}\right]_t^1 = \pi\left(\frac{1}{\sqrt{2}}-\sqrt{2}+\sqrt{t^2+1}-\frac{t}{\sqrt{2}}\right),
\end{aligned}$$

which is convergent for $t\to 0+$.

The improper space integral exists, and its value is

$$\int_A \frac{z^3}{(x^2+y^2+z^4)^{\frac{3}{2}}}\,d\Omega = \pi\left(\frac{1}{\sqrt{2}}-\sqrt{2}+1\right) = \frac{\pi}{2}(\sqrt{2}-2\sqrt{2}+2) = \frac{\pi}{2}(2-\sqrt{2}).$$

3) The set A lies in the first octant, thus the integrand is defined and positive everywhere in A. However, A is unbounded. First note that we have in spherical coordinates,

$$xyz = r^3\sin^2\theta\,\cos\theta\,\cos\varphi\,\sin\varphi,$$

which does not look too promising to put into the denominator. Note instead that we have the following estimates in A,

$$1 \le 1+xyz \le 1+r^3.$$

When we choose the following truncation in spherical coordinates

$$A_R = \left\{(r,\theta,\varphi)\;\middle|\; 0\le r\le R,\,\theta\in\left[\frac{\pi}{3},\frac{\pi}{2}\right],\,\varphi\in\left[\frac{\pi}{6},\frac{\pi}{3}\right]\right\},\quad R>0,$$

we get the following estimate,

$$\begin{aligned}
\int_{A_R}\frac{1}{1+xyz}\,d\Omega &\ge \int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left\{\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\left\{\int_0^R \frac{1}{1+r^3}\,r^2\sin\theta\,dr\right\}d\theta\right\}d\varphi \\
&= \frac{\pi}{6}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\sin\theta\,d\theta\cdot\int_0^R\frac{r^2}{1+r^3}\,dr = \frac{\pi}{6}\left[-\cos\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}\left[\frac{1}{3}\,\ln(1+r^2)\right]_0^R = \frac{\pi}{36}\,\ln(1+R^2).
\end{aligned}$$

Thus we conclude that

$$\lim_{R\to+\infty}\int_{A_R}\frac{1}{1+xyz}\,d\Omega = +\infty,$$

and the improper space integral does not exist.

4) The domain is bounded and the integrand is positive in A, with the exception of $(0,0,0)$, where it is not defined. Put in semi-polar coordinates,

$$A_t = \{(\varrho,\varphi,z) \mid t \le \varrho \le h,\ \varphi \in [0,2\pi],\ \varrho \le z \le h\}, \quad 0 < t < h.$$

When $h = 1$ the projection B_t of A_t is the same as in **Example 6.1.2**.

Then we compute,

$$\begin{aligned}\int_{A_t} \frac{1}{z^2}\,d\Omega &= \int_0^{2\pi}\left\{\int_t^h\left\{\int_\varrho^h \frac{1}{z^2}\,dz\right\}\varrho\,d\varrho\right\}d\varphi = 2\pi\int_t^h\left[-\frac{1}{z}\right]_{z=\varrho}^h \varrho\,d\varrho = 2\pi\int_t^h\left(\frac{1}{\varrho}-\frac{1}{h}\right)\varrho\,d\varrho \\ &= 2\pi\int_t^h\left(1-\frac{1}{h}\,\varrho\right)d\varrho = \left[\varrho-\frac{\varrho^2}{2h}\right]_{\varrho=t}^h = 2\pi\left[h-\frac{h^2}{2h}-t+\frac{t^2}{2h}\right] \to \pi h \quad \text{for } t \to 0+.\end{aligned}$$

Hence, the improper space integral exists and its value is given by

$$\int_A \frac{1}{z^2}\,d\Omega = \pi h.$$

5) The integrand is defined all over $\mathbb{R}^3$, and *if* the improper integral exists, then its value must for symmetric reasons necessarily be 0.

The integrand is ≥ 0 in the upper half space. Hence we define in semi-polar coordinates the following truncation,

$$A_R = \{(\varrho, \varphi, z) \mid 0 \leq \varrho \leq R,\, \varphi \in [0, 2\pi],\, 0 \leq z \leq R\}, \quad R > 0.$$

We get by integration over this set

$$\begin{aligned}\int_{A_R} z\, \exp\left(-(x^2+y^2+z^2)\right) d\Omega &= \int_0^R z\, e^{-z^2} dz \cdot \int_0^{2\pi} \left\{\int_0^R e^{-\varrho^2} \varrho\, d\varrho\right\} d\varphi \\ &= 2\pi \left\{\int_0^R t\, e^{-t^2} dt\right\}^2 = 2\pi \left\{\left[-\frac{1}{2} e^{t^2}\right]_0^R\right\}^2 = \frac{\pi}{2}\left(1 - e^{-R^2}\right)^2,\end{aligned}$$

which clearly converges for $R \to +\infty$.

Similarly the improper space integral exists over the lower half space, and the improper space integral exists. Due the to symmetry, the value must be

$$\int_A z\, \exp\left(-(x^2+y^2+z^2)\right) d\Omega = 0.$$

6) The integrand is positive all over $\mathbb{R}^3$, so we choose the truncation

$$[-a, a] \times [-b, b] \times [-c, c], \qquad a,\, b,\, c > 0.$$

After the integration over this set we shall continue by letting a, b and c tend to $+\infty$ independently. This procedure can be shortened by using a well-known example.

We get by the change of variables,

$$\xi = \sqrt{2}\, x, \qquad \eta = \sqrt{3}\, y, \qquad \zeta = \sqrt{6}\, z,$$

that

$$\begin{aligned}&\int_{[-a,a]\times[-b,b]\times[-c,c]} \exp\left(-2x^2 - 3y^2 - 6z^2\right) d\Omega \\ &= \int_{-a}^{a} \exp(-2x^2) dx \cdot \int_{-b}^{b} \exp(-3y^2) dy \cdot \int_{-c}^{c} \exp(-6z^2) dz \\ &= 2^3 \int_0^a \exp(-2x^2) dx \cdot \int_0^b \exp(-3y^2) dy \cdot \int_0^c \exp(-6z^2) dz \\ &= 8 \cdot \frac{1}{\sqrt{2}} \int_0^{a\sqrt{2}} \exp(-\xi^2) d\xi \cdot \frac{1}{\sqrt{3}} \int_0^{b\sqrt{3}} \exp(-\eta^2) d\eta \cdot \frac{1}{\sqrt{6}} \int_0^{c\sqrt{6}} \exp(-\zeta^2) d\zeta.\end{aligned}$$

Each of the three integrals is convergent by taking the limit and the value is

$$\int_0^{+\infty} \exp(-t^2)\, dt = \frac{\sqrt{\pi}}{2}.$$

Hence we conclude that the improper space integral is convergent and its value is

$$\int_{\mathbb{R}^3} \exp(-2x^2 - 3y^2 - 6z^2) d\Omega = 8 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{6}} \cdot \left(\frac{\sqrt{\pi}}{2}\right)^3 = \frac{\pi\sqrt{\pi}}{6}.$$

Example 6.2 *In each of the following cases is given a space integral including a parameter $\alpha \in \mathbb{R}$. The integral is improper for some or all values of the parameter α.*
Let M_C and $M_D = \mathbb{R} \setminus M_C$ be sets of numbers such that the integral is convergent (or proper) for $\alpha \in M_C$ and divergent for $\alpha \in M_C$. Find in each of the following cases M_C and M_D as well as the value of the integral for $\alpha \in M_C$.

1) *The space integral $\int_A (x^2+y^2+z^2)^{-\alpha} d\Omega$, where the domain of integration A is given by $x^2+y^2+z^2 \leq 1$.*

2) *The space integral $\int_A (x^2+y^2+z^2)^{-\alpha} d\Omega$, where the domain of integration A is given by $x^2+y^2+z^2 \geq 1$.*

3) *The space integral $\int_A |z| \exp\left(-\alpha(x^2+y^2+z^2)\right) d\Omega$, where the domain of integration is given by $A = \mathbb{R}^3$.*

4) *The space integral $\int_A \left\{z + \exp\left(-\alpha(x^2+y^2+z^2)\right)\right\} d\Omega$, where the domain of integration is given by $A = \mathbb{R}^3$.*

5) *The space integral $\int_{A(\alpha)} \dfrac{1}{x^2+y^2}\, d\Omega$, where the domain of integration $A(\alpha)$ is described by*

$$z^{|\alpha|} \leq x^2+y^2 \leq 1, \qquad 0 \leq z \leq 1.$$

A Improper space integrals with a parameter.

D Describe why the space integral is improper. Analyze and compute the space integral based on the classification.

I 1) The domain of integration A is the closed unit ball. The integrand is positive in $A \setminus \{(0,0,0)\}$, and it is not defined at the point $(0,0,0)$.

The formulation gives of hint of an application of spherical coordinates. We choose the truncation

$$A_R = \{(r,\theta,\varphi) \mid R \leq r \leq 1,\, \theta \in [0,\pi],\, \varphi \in [0,2\pi]\}, \quad 0 < R < 1,$$

i.e. we remove a small ball of radius R from A. When we integrate over A_R we get

$$\begin{aligned}\int_{A_R} (x^2+y^2+z^2)^{-\alpha} d\Omega &= \int_0^{2\pi} \left\{ \int_0^{\pi} \left\{ \int_R^1 r^{-2\alpha} \cdot r^2 \sin\theta\, dr \right\} d\theta \right\} d\varphi \\ &= 2\pi \cdot [-\cos\theta]_0^{\pi} \cdot \int_R^1 r^{2(1-\alpha)} dr = 4\pi \int_R^1 r^{2(1-\alpha)} dr.\end{aligned}$$

It is well-known that the limit exists for $R \to 0+$, if and only if $2(1-\alpha) > -1$, i.e. if and only if $\alpha < \dfrac{3}{2}$. Hence,

$$M_C = \left\{ \alpha \;\middle|\; \alpha < \frac{3}{2} \right\} \quad \text{og} \quad M_D = \left\{ \alpha \;\middle|\; \alpha \geq \frac{3}{2} \right\}.$$

If $\alpha \in M_C$, i.e. if $\alpha < \dfrac{3}{2}$, we get the value of the integral by taking the limit,

$$\int_A (x^2+y^2+z^2)^{-\alpha} d\Omega = \lim_{R \to 0+} 4\pi \int_R^1 r^{2(1-\alpha)} dr = 4\pi \left[\frac{r^{3-2\alpha}}{3-2\alpha} \right]_0^1 = \frac{4\pi}{3-2\alpha}.$$

2) The integrand is the same as in 1). The domain of integration, however, has been replaced by the complementary set of the unit ball. The integrand is defined and continuous everywhere in the unbounded set A. We choose the truncation

$$A_R = \{(r,\theta,\varphi) \mid 1 \le r \le R,\, \theta \in [0,\pi],\, \varphi \in [0,2\pi]\}, \quad R > 1.$$

Using the same computation as in 1), i.e. first integrate with respect to θ and φ, then

$$\int_{A_R} (x^2+y^2+z^2)^{-\alpha} d\Omega = 4\pi \int_1^R r^{2(1-\alpha)} dr.$$

The limit exists for $R \to +\infty$, if and only if $2(1-\alpha) < -1$, i.e. if and only if $\alpha > \frac{3}{2}$. It follows that

$$M_C = \left\{ \alpha \;\middle|\; \alpha > \frac{3}{2} \right\} \quad \text{og} \quad M_D = \left\{ \alpha \;\middle|\; \alpha \le \frac{3}{2} \right\}.$$

Let $\alpha \in M_C$, i.e. $\alpha > \frac{3}{2}$. Then we get

$$\int_A (x^2+y^2+z^2)^{-\alpha} d\Omega = \lim_{R\to+\infty} 4\pi \int_1^R r^{2(1-\alpha)} dr = \lim_{R\to+\infty} 4\pi \left[-\frac{r^{-(2\alpha-3)}}{2\alpha-3} \right]_1^R = \frac{4\pi}{2\alpha-3}.$$

3) The space integral clearly does not exist when $\alpha \le 0$, because the integrand then tends "uniformly" towards $+\infty$ for e.g. $|z| \ge 1$. Thus,

$$M_D \supseteq \{\alpha \mid \alpha \le 0\}.$$

Then let $\alpha > 0$. The integrand is ≥ 0 in $\mathbb{R}^3$, so we put in semi-polar coordinates [cf. **Example 6.1.5**],

$$A_R = \{(\varrho,\varphi,z) \mid 0 \le \varrho \le R,\, \varphi \in [0,2\pi],\, -R \le z \le R\}, \quad R > 0.$$

When we integrate over A_R, we get

$$\begin{aligned}
\int_{A_R} |z| \exp\left(-\alpha(x^2+y^2+z^2)\right) d\Omega &= \int_0^{2\pi} \left\{ \int_0^R \left\{ 2\int_0^R z \exp\left(-\alpha(\varrho^2+z^2)\right) dz \right\} \varrho\, d\varrho \right\} d\varphi \\
&= 2\cdot 2\pi \int_0^R e^{-\alpha\varrho^2} \varrho\, d\varrho \cdot \int_0^R e^{-\alpha z^2} z\, dz = \pi \left\{ \int_0^R e^{-\alpha t^2} \cdot 2t\, dt \right\}^2 = \pi \left\{ \left[-\frac{e^{-\alpha t^2}}{\alpha} \right]_0^R \right\}^2 \\
&= \frac{\pi}{\alpha^2} \left(1 - e^{-\alpha R^2} \right)^2,
\end{aligned}$$

which clearly converges for $R \to +\infty$, because $\alpha > 0$. We conclude that

$$M_C = \{\alpha \mid \alpha > 0\} \quad \text{and} \quad M_D = \{\alpha \mid \alpha \le 0\}.$$

Let $\alpha \in M_C$, i.e. $\alpha > 0$. Then

$$\int_A |z| \exp\left(-\alpha(x^2+y^2+z^2)\right) d\Omega = \lim_{R\to+\infty} \frac{\pi}{\alpha^2} \left(1 - e^{-\alpha R^2} \right)^2 = \frac{\pi}{\alpha^2}.$$

4) The space integral is divergent for every $\alpha \in \mathbb{R}$, i.e. $M_C = \emptyset$ and $M_D = \mathbb{R}$. One may in semi-polar coordinates put

$$A_R = \{(\varrho, \varphi, z) \mid 0 \le \varrho \le R,\ \varphi \in [0, 2\pi],\ 0 \le z \le R\}, \quad R > 0.$$

Then

$$\int_{A_R} \left\{z + \exp(-\alpha(x^2+y^2+z^2))\right\} d\Omega \ge \int_{A_R} z\, d\Omega = \pi R^2 \int_0^R z\, dz = \frac{\pi}{2}\, R^4,$$

which clearly goes to $+\infty$ for $R \to +\infty$.

5) The set $A(\alpha)$ is bounded, and the integrand is not defined at $(0,0,0)$. The integrand is positive in the remaining part of the domain of integration.

Consider for a fixed $R \in]0,1[$ (in semi-polar coordinates) the truncated domain

$$A_R = \begin{cases} \{(\varrho, \varphi, z) \mid R \le \varrho \le 1,\ \varphi \in [0, 2\pi],\ 0 \le z \le \varrho^{\frac{2}{|\alpha|}}\}, & \alpha \neq 0, \\ \{(\varrho, \varphi, z) \mid \varrho = 1,\ \varphi \in [0, 2\pi],\ 0 \le z \le 1\}, & \alpha = 0. \end{cases}$$

a) If $\alpha = 0$, then $(0,0,0)$ does not belong to $A(0)$, and $A(0)$ is a cylindric *surface*. A *space integral* over a smooth *surface* is 0. We therefore conclude that $0 \in M_C$ and

$$\int_{A(0)} \frac{1}{x^2+y^2}\, d\Omega = 0.$$

b) If $\alpha \neq 0$, we get by integration over the truncated domain that

$$\int_{A_R} \frac{1}{x^2+y^2}\, d\Omega = \int_0^{2\pi} \left\{ \int_R^1 \frac{1}{\varrho^2} \left\{ \int_0^{\varrho^{2/|\alpha|}} dz \right\} \varrho\, d\varrho \right\} d\varphi$$

$$= 2\pi \int_R^1 \varrho^{2/|\alpha|-1} dz\, d\varrho = 2\pi \left[\frac{|\alpha|}{2}\, \varrho^{2/|\alpha|} \right]_R^1 = \pi|\alpha| \left\{ 1 - R^{2/|\alpha|} \right\}.$$

From $\dfrac{2}{|\alpha|} > 0$ follows that this expression converges for $R \to 0+$, thus $\alpha \in M_C$, and

$$(2) \quad \int_{A(\alpha)} \frac{1}{x^2+y^2}\, d\Omega = \pi|\alpha| \qquad \text{for } \alpha \neq 0.$$

By comparison of the two cases above we see that (2) is also valid for $\alpha = 0$, hence

$$M_C = \mathbb{R} \qquad \text{and} \qquad M_D = \emptyset,$$

and

$$\int_{A(\alpha)} \frac{1}{x^2+y^2}\, d\Omega = \pi|\alpha| \qquad \text{for } \alpha \in M_C = \mathbb{R}.$$

Example 6.3 *When we rotate the meridian curve $\mathcal{M}$ given by the parametric description*

$$\varrho = a\,\cos t, \quad z = a\{\ln(1+\sin t) - \ln\cos t - \sin t\}, \qquad t \in \left[0, \frac{\pi}{2}\right[,$$

we obtain a surface of revolution (half of the pseudosphere), which together with a disc in the XY-plane bound a body of revolution Ω. Both $\mathcal{M}$ and Ω go to infinity along the positive part of the Z-axis. Find $\dfrac{dz}{dt}$, and set up an expression by an integral for the volume oft that part of Ω, which corresponds to $t \in [0,T]$, where $T < \dfrac{\pi}{2}$.
Then find the volume of Ω by letting $T \to \dfrac{\pi}{2}$.

A Volume of an infinite body of revolution; improper space integral.

D Sketch the meridian curve. Then follow the guidelines.

I First calculate

$$\begin{aligned} \frac{dz}{dt} &= a\left\{ \frac{\cos t}{1+\sin t} + \frac{\sin t}{\cos t} - \cos t \right\} = a\left\{ \frac{\cos t(1-\sin t)}{1-\sin^2 t} + \frac{\sin t}{\cos t} - \cos t \right\} \\ &= a\left\{ \frac{1-\sin t+\sin t}{\cos t} - \cos t \right\} = a \cdot \frac{1-\cos^2 t}{\cos t} = a\,\frac{\sin^2 t}{\cos t}. \end{aligned}$$

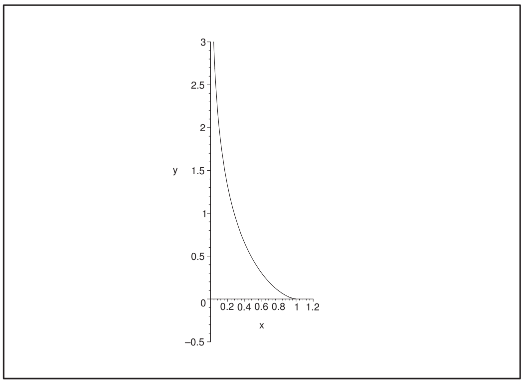

Figure 47: The meridian curve $\mathcal{M}$ in the PZ half plane.

Let Ω_T be that part of the body of revolution which corresponds to $t \in [0,T]$ by the parametric description. The plane $z = z(t)$, $t \in [0,T]$, cuts a disc out of the body of revolution of the area $\pi a^2 \cos^2 t$, thus

$$\begin{aligned}\text{vol}(\Omega_T) &= \int_0^T \pi a^2 \cos^2 t \cdot \frac{dz}{dt}\,dt = \pi a^3 \int_0^T \cos^2 \cdot \frac{\sin^2 t}{\cos t}\,dt \\ &= \pi a^3 \int_0^T \sin^2 t\, \cos t\,dt = \frac{\pi a^3}{3}\,\left[\sin^3 t\right]_0^T = \frac{\pi a^3}{3}\,\sin^3 T.\end{aligned}$$

The volume of "half" of the pseudosphere is then found by taking the limit,

$$\text{vol}(\Omega) = \lim_{T \to \frac{\pi}{2}-} \text{vol}(\Omega_T) = \frac{\pi a^3}{3}.$$

Example 6.4 *There is given a curve $\mathcal{K}$ in the (x,y)-plane of the equation*

$$y^2(a-x) = x^3.$$

1) Show that the curve lies in the strip $[0,a] \times \mathbb{R}$, and that the line $x = a$ is an asymptote for $\mathcal{K}$. Sketch $\mathcal{K}$.

2) By revolving $\mathcal{K}$ with the asymptote as axis we get a body of revolution Ω, which stretches into infinity. Prove that Ω has the volume $\dfrac{1}{4}\,\pi^2 a^3$.

A A body of revolution given by a meridian curve; improper space integral.

D Follow the guideline.

I 1) Clearly, the curve is symmetric with respect to the X-axis, and $x = 0$ for $y = 0$. If $x \in [0,a[$, then

$$y^2 = \frac{x^2}{a-x}, \qquad \text{i.e. } y = \pm x\sqrt{\frac{x}{a-x}},$$

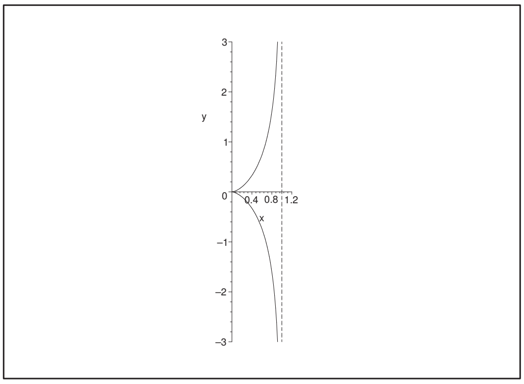

Figure 48: The curve $\mathcal{K}$.

because $\dfrac{x}{a-x} > 0$.
If $x \to a-$, then $|y| \to +\infty$, thus $x = a$ is an asymptote.

If $x = a$ we have no solution ($y^2 \cdot 0 \neq a^3$).
If $x > a$, then $a - x < 0$ and $x^3 > 0$. There is no solution, because $y^2 \geq 0$.
Similarly, $a - x > 0$ and $x^3 < 0$, when $x < 0$, so there in no solution here either, and we have proved 1).

2) Let $x \in [0, a[$. Of symmetric reasons it suffices to consider $y \geq 0$, i.e.

$$y = x\sqrt{\frac{x}{a-x}} = \sqrt{\frac{x^3}{a-x}} = x^{\frac{3}{2}}(a-x)^{-\frac{1}{2}}.$$

Then

$$\frac{dy}{dx} = \frac{3}{2}\,x^{\frac{1}{2}}(a-x)^{-\frac{1}{2}} + \frac{1}{2}\,x^{\frac{3}{2}}(a-x)^{-\frac{3}{2}} = \frac{1}{2}\,\frac{x^{\frac{1}{2}}}{(a-x)^{\frac{3}{2}}}\{3(a-x)+x\} = \frac{1}{2}\sqrt{\frac{x}{(a-x)^3}}\cdot(3a-2x).$$

The area of the circle of rotation C_x (with respect to the line $x = a$) at the height $y(x)$ is $\pi(x-a)^2$. If we truncate at the height $y(x_0)$ corresponding to some $x_0 \in [0, a[$, we get the corresponding volume (notice the symmetry with respect to $y = 0$)

$$\begin{aligned} 2\int_0^{x_0} \text{area}(C_x)\cdot\frac{dy}{dx}\,dx &= 2\int_0^{x_0} \pi(x-a)^2\cdot\frac{1}{2}\sqrt{\frac{x}{(a-x)^3}}\cdot(3a-2x)\,dx \\ &= \pi\int_0^{x_0} \sqrt{x(a-x)}\cdot(3a-2x)\,dx. \end{aligned}$$

We have clearly convergency for $x_0 \to a-$, thus the total volume is by the change of variable

$t = x - \frac{a}{2}$ given by

$$
\begin{aligned}
\pi \int_0^a \sqrt{x(a-x)}\cdot(3a-2x)\,dx &= \pi \int_0^a \sqrt{\left(\frac{a}{2}\right)^2 - \left(x-\frac{a}{2}\right)^2}\cdot\left\{2a - 2\left(x-\frac{a}{2}\right)\right\}dx \\
&= \pi \int_{-\frac{a}{2}}^{\frac{a}{q}2} \sqrt{\left(\frac{a}{2}\right) - t^2}\cdot\{2a-2t\}dt = 2\pi a \int_{-\frac{a}{2}}^{\frac{a}{2}} \sqrt{\left(\frac{a}{2}\right)^2 - t^2}\,dt + 0 \\
&= 2\pi a \cdot \frac{1}{2}\,\pi\left(\frac{a}{2}\right)^2 = \frac{\pi^2 a^3}{4},
\end{aligned}
$$

because it again follows by the symmetry (the integrand is an odd function) that

$$
-\pi \int_{-\frac{a}{2}}^{\frac{a}{2}} \sqrt{\left(\frac{a}{2}\right)^2 - t^2}\cdot 2t\,dt = 0.
$$

Finally,

$$
\int_{-\frac{a}{2}}^{\frac{a}{2}} \sqrt{\left(\frac{a}{2}\right)^2 - t^2}\,dt = \frac{1}{2}\,\pi\left(\frac{a}{2}\right)^2
$$

is the area of the half disc in the upper half plane of radius $\frac{a}{2}$ and centrum $\left(\frac{a}{2},0\right)$.

Example 6.5 *Given in the meridian half plane a curve of the equation*

$$\varrho = \frac{a^3}{a^2+z^2}, \qquad z \in \mathbb{R}.$$

The curve is rotated around the vertical Z-axis, and we get a body of revolution Ω, which stretches into infinity. Prove that Ω has the volume $\dfrac{\pi^2 a^3}{2}$.

A Volume of an infinite body of revolution; improper space integral.

D Sketch the meridian curve. Set up the improper space integral and compute by first truncate to a bounded domain.

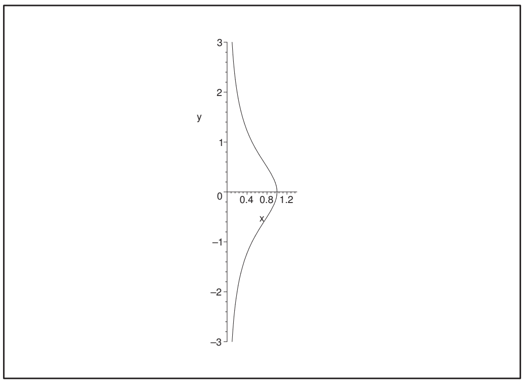

Figure 49: The meridian curve in the PZ half plane.

I The curve is clearly symmetric about the P-axis, and it suffices to consider $z \geq 0$.

THE SLICING METHOD. We cut a disc $C(z)$ out of the body of revolution of radius ϱ at the height z. Then

$$\text{area}(C(z)) = \pi\varrho(z)^2 = \frac{\pi a^6}{(a^2+z^2)^2}.$$

Define the truncation Ω_k by

$$\Omega_k = \{(x,y,z) \in \Omega \mid |z| \leq ka\}, \qquad k > 0.$$

Then by integration over Ω_k,

$$\text{vol}(\Omega_k) = 2\int_0^{ka} \text{area}(C(z))\,dz = 2\pi a^6 \int_0^{ka} \frac{1}{(a^2+z^2)^2}\,dz = 2\pi a^3 \int_0^k \frac{dt}{(1+t^2)^2},$$

which clearly is convergent for $k \to +\infty$. The improper integral exists, and we get by the substitution $t = \tan u$,

$$\begin{aligned}\mathrm{vol}(\Omega) &= 2\pi a^3 \int_0^{+\infty} \frac{1}{(1+t^2)^2}\,dt = 2\pi a^3 \int_0^{\frac{\pi}{2}} \frac{1}{(1+\tan^2 u)^2}\,(1+\tan^2 u)\,du \\ &= 2\pi a^3 \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^2 u}\,du = 2\pi a^3 \int_0^{\frac{\pi}{2}} \cos^2 u\,du \\ &= 2\pi a^3 \cdot \frac{1}{2} \int_0^{\frac{\pi}{2}} (\cos^2 u + \sin^2 u)\,du = \pi a^3 \cdot \frac{\pi}{2} = \frac{\pi^2 a^3}{2}.\end{aligned}$$

Example 6.6 *Let A be the infinite half cylinder, which in semi-polar coordinates is bounded in the following way*

$$0 \le z < +\infty, \qquad 0 \le \varrho \le a, \qquad 0 \le \varphi \le 2\pi.$$

Prove that the improper space integral

$$\int_A \frac{z^2(a-x)}{\{(a-z)^2+y^2+z^2\}^2}\,d\Omega$$

is convergent, and find its value.

A Improper space integral in semi-polar coordinates.

D Find the possible points where the integrand is not defined and truncate around them as well as truncate infinity. Apparently, the computations are "easiest" in rectangular coordinates.

I The integrand is not defined at the point $(a,0,0)$, and it is positive elsewhere in $A\setminus(a,0,0)$. Choose the truncation

$$A_{\varepsilon,T} = \{(\varrho,\varphi,z) \mid 0 \le \varrho \le a,\ \varphi \in [0,2\pi],\ \varepsilon \le z \le T\}, \quad 0 < \varepsilon < T < +\infty.$$

Denote the projection of A and $A_{\varepsilon,T}$ onto the XY-plane by B, thus

$$B = \{(x,y) \mid x^2+y^2 \le a^2\} = \{(\varrho,\varphi) \mid 0 \le \varrho \le a,\ \varphi \in [0,2\pi]\}.$$

By reduction of the space integral over $A_{\varepsilon,T}$ we get

$$\int_{A_{\varepsilon,T}} \frac{z^2(a-x)}{\{(a-z)^2+y^2+z^2\}^2}\,d\Omega = \int_\varepsilon^T z^2 \left\{\int_B \frac{a-x}{\{(a-x)^2+y^2+z^2\}^2}\,dx\,dy\right\} dz.$$

For every fixed $z \in [\varepsilon, T]$,

$$\begin{aligned}&\int_B \frac{a-x}{\{(a-x)^2+y^2+z^2\}^2}\,dx\,dy = \int_{-a}^a \left\{\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} \frac{a-x}{\{(a-x)^2+y^2+z^2\}^2}\,dx\right\} dy \\ &= 2\int_0^a \left[+\frac{1}{2}\cdot\frac{1}{(a-x)^2+y^2+z^2}\right]_{x=-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} dy \\ &= \int_0^a \left\{\frac{1}{(a-\sqrt{a^2-y^2})^2+y^2+z^2} - \frac{1}{(a+\sqrt{a^2-y^2})^2+y^2+z^2}\right\} dy \\ &= \int_0^a \left\{\frac{1}{2a^2-2a\sqrt{a^2-y^2}+z^2} - \frac{1}{2a^2+2a\sqrt{a^2-y^2}+z^2}\right\} dy,\end{aligned}$$

hence by insertion and interchanging the order of integration,

$$\begin{aligned}
&\int_{A_{\varepsilon,T}} \frac{z^2(a-x)}{\left\{(a-x)^2+y^2+z^2\right\}^2}\,d\Omega\\
&\quad= \int_\varepsilon^T \left\{ \int_0^a \left\{ \frac{z^2}{2a^2-2a\sqrt{a^2-y^2}+z^2} - \frac{z^2}{2a^2+2a\sqrt{a^2-y^2}} \right\} dy \right\} dz\\
&\quad= \int_0^a \left\{ \int_\varepsilon^T \left\{ \frac{(2a^2\ -2a\sqrt{a^2-y^2}+z^2)-(2a^2-2a\sqrt{a^2\ -y^2})}{2a^2+z^2-2a\sqrt{a^2-y^2}} \right.\right.\\
&\qquad\qquad \left.\left. - \frac{(2a^2+2a\sqrt{a^2-y^2}+z^2)-(2a^2+2a\sqrt{a^2-y^2})}{2a^2+z^2+2a\sqrt{a^2-y^2}} \right\} dz \right\} dy\\
&\quad= \int_0^a \left\{ \int_\varepsilon^T \frac{2a^2+2a\sqrt{a^2-y^2}}{2a^2+2a\sqrt{a^2-y^2}+z^2}\,dz \right\} dy - \int_0^a \left\{ \int_\varepsilon^T \frac{2a^2-2a\sqrt{a^2-y^2}}{2a^2-2a\sqrt{a^2-y^2}+z^2}\,dz \right\} dy\\
&\quad= \int_0^a \left[\sqrt{2a^2+2a\sqrt{a^2-y^2}}\,\operatorname{Arctan}\left(\frac{z}{\sqrt{2a^2+2a\sqrt{a^2-y^2}}} \right) \right]_{z=\varepsilon}^T dy\\
&\qquad\quad - \int_0^a \left[\sqrt{2a^2-2a\sqrt{a^2-y^2}}\,\operatorname{Arctan}\left(\frac{z}{\sqrt{2a^2-2a\sqrt{a^2-y^2}}} \right) \right]_{z=\varepsilon}^T dy.
\end{aligned}$$

For fixed $y>0$ (in fact also for "$y=0$") the sum of the integrands converges for $\varepsilon \to 0+$ and $T \to +\infty$ towards

$$\frac{\pi}{2}\left\{ \sqrt{2a^2+2a\sqrt{a^2-y^2}} - \sqrt{2a^2-2a\sqrt{a^2-y^2}} \right\},$$

thus we get by these limits and the substitution $y = a\sin t$, $t \in \left[0,\frac{\pi}{2}\right]$, that the improper space integral is convergent with the value

$$\begin{aligned}
&\int_A \frac{z^2(a-x)}{\left\{(a-x)^2+y^2+z^2\right\}^2}\,d\Omega\\
&\quad= \lim_{\delta\to 0+} \int_\delta^{\frac{\pi}{2}} \frac{\pi}{2}\left\{ \sqrt{2a^2+2a^2\cos t} - \sqrt{2a^2-2a^2\cos t} \right\} \cdot a\cos t\,dt\\
&\quad= \frac{\pi}{2}\sqrt{2}\cdot a^2 \int_0^{\frac{\pi}{2}} \left\{ \sqrt{1+\cos t} - \sqrt{1-\cos t} \right\} \cos t\,dt\\
&\quad= \pi a^2 \int_0^{\frac{\pi}{2}} \left\{ \cos\frac{t}{2} - \sin\frac{t}{2} \right\} \left(\cos^2\frac{t}{2} - \sin^2\frac{t}{2} \right) dt\\
&\quad= \pi a^2 \int_0^{\frac{\pi}{2}} \left(1 - 2\sin^2\frac{t}{2} \right) \cos\frac{t}{2}\,dt - \pi a^2 \int_0^{\frac{\pi}{2}} \left(2\cos^2\frac{t}{2} - 1 \right) \sin\frac{t}{2}\,dt\\
&\quad= \pi a^2 \cdot 2 \left[\sin\frac{t}{2} - \frac{2}{3}\sin^3\frac{t}{2} \right]_0^{\frac{\pi}{2}} - \pi a^2 \cdot 2 \left[-\frac{2}{3}\cos^3\frac{t}{2} + \cos\frac{t}{2} \right]_0^{\frac{\pi}{2}}\\
&\quad= 2\pi a^2 \left\{ \frac{1}{\sqrt{2}} - \frac{2}{3}\cdot\frac{1}{2\sqrt{2}} + \frac{2}{3}\cdot\frac{1}{2\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{2}{3} + 1 \right\} = \frac{2\pi a^2}{3}.
\end{aligned}$$

Example 6.7 *It is well-known that we by considering the improper plane integral*

$$\int_{\mathbb{R}^2} \exp\left(-x^2-y^2\right)\, dS$$

can derive the formula

$$\int_{-\infty}^{\infty} \exp\left(-u^2\right)\, du = \sqrt{\pi}.$$

1) Find the integral

$$\int_{-\infty}^{\infty} \exp\left(-\frac{u^2}{a}\right) du.$$

2) Find by considering the improper space integral

$$\int_{\mathbb{R}^3} \exp\left(-x^2-y^2-z^2\right)\, d\Omega,$$

the integral

$$\int_{-\infty}^{\infty} u^2 \exp\left(-\frac{u^2}{a}\right) du.$$

3) Let p be a constant in the interval $]-1,1[$. Compute the plane integral

$$\int_{\mathbb{R}^2} \exp\left(-\frac{x^2-2pxy+y^2}{2(1-p^2)}\right) dS$$

by first integration over the parallelogram $B(T)$ with the vertices

$$(-T-pT,-T), \quad (T-pT,-T), \quad (T+pT,T), \quad (-T+pT,T),$$

and then by putting $x = py+t$.

4) Prove that the improper plane integral

$$\int_{\mathbb{R}^2} xy\, \exp\left(-\frac{x^2-2py+y^2}{2(1-p^2)}\right) dS$$

is convergent. Then find its value by using the same method as in the previous question.

A Improper space and plane integrals.

D Either follow the given guidelines, or (which is possible here) design an alternative.

I First notice that if we put

$$B_R = \{(x,y) \mid x^2+y^2 \le R^2\},$$

then we get by using polar coordinates,

$$\int_{B_R} \exp\left(-x^2-y^2\right)\, dS = 2\pi \int_0^R \exp\left(-\varrho^2\right)\, \varrho\, d\varrho = \pi\left\{1-\exp\left(-R^2\right)\right\}.$$

The integrand is positive, hence by taking the limit $R \to +\infty$,

$$\int_{\mathbb{R}^2} \exp\left(-x^2-y^2\right)\, dS = \lim_{R\to+\infty} \int_{B_R} \exp\left(-x^2-y^2\right)\, dS = \pi,$$

thus

$$\pi = \int_{\mathbb{R}^2} \exp\left(-x^2-y^2\right)\, dS = \left\{\int_{\mathbb{R}} \exp\left(-x^2\right)\, dx\right\}\left\{\int_{\mathbb{R}} \exp\left(-y^2\right)\, dy\right\} = \left\{\int_{-\infty}^{+\infty} \exp\left(-u^2\right)\, du\right\}^2,$$

and we get

$$(3)\quad \int_{-\infty}^{+\infty} \exp\left(-u^2\right)\, du = \sqrt{\pi}.$$

1) It follows from (3) with $u = \dfrac{t}{\sqrt{a}}$ that

$$\sqrt{\pi} = \int_{-\infty}^{+\infty} \exp\left(-u^2\right)\, du = \int_{-\infty}^{+\infty} \exp\left(-\frac{t^2}{a}\right) \frac{1}{\sqrt{a}}\, dt,$$

hence

$$\int_{-\infty}^{+\infty} \exp\left(-\frac{u^2}{a}\right) du = \sqrt{a\pi}.$$

2) Then by a partial integration and an application of the result of 1),

$$\begin{aligned}\int_{-\infty}^{+\infty} u^2 \exp\left(-\frac{u^2}{a}\right) du &= \frac{a}{2}\int_{-\infty}^{+\infty} u \cdot 2\,\frac{u}{a} \cdot \exp\left(-\frac{u^2}{a}\right) du \\ &= \frac{a}{2}\left[-u\,\exp\left(-\frac{u^2}{a}\right)\right]_{-\infty}^{+\infty} + \frac{a}{2}\int_{-\infty}^{+\infty} \exp\left(-\frac{u^2}{a}\right) du = \frac{1}{2}\,a\sqrt{2\pi}.\end{aligned}$$

Alternatively put $\Omega_R = \{(x,y,z) \mid x^2+y^2+z^2 \leq R^2\}$. Then

$$\begin{aligned}\int_{\Omega_R} \exp\left(-x^2-y^2-z^2\right) d\Omega &= \int_0^{2\pi}\left\{\int_0^{\pi}\left\{\int_0^R \exp\left(-r^2\right) r^2 \sin\theta\, dr\right\} d\theta\right\} d\varphi \\ &= 2\pi \cdot 2\int_0^R r^2 \exp\left(-r^2\right)\, dr,\end{aligned}$$

hence

$$\begin{aligned}\pi\sqrt{\pi} &= \left\{\int_{-\infty}^{+\infty} \exp\left(-u^2\right)\, du\right\}^3 = \int_{\mathbb{R}^3} \exp\left(-x^2-y^2-z^2\right)\, d\Omega \\ &= \lim_{R\to+\infty} \int_{\Omega_R} \exp\left(-x^2-y^2-z^2\right)\, d\Omega \\ &= 4\pi\int_0^{+\infty} r^2 \exp\left(-r^2\right)\, dr = 2\pi\int_{-\infty}^{+\infty} u^2 \exp\left(-u^2\right)\, du,\end{aligned}$$

and thence

$$\int_{-\infty}^{+\infty} u^2 \exp\left(-u^2\right)\, du = \frac{\sqrt{\pi}}{2}.$$

Then we get by the change of variable $u \to \dfrac{u}{\sqrt{a}}$,

$$\int_{-\infty}^{+\infty} \frac{u^2}{a}\, \exp\left(-\frac{u^2}{a}\right) \cdot \frac{1}{\sqrt{a}}\, du = \frac{\sqrt{\pi}}{2},$$

i.e.

$$\int_{-\infty}^{+\infty} u^2 \exp\left(-\frac{u^2}{a}\right) du = \frac{1}{2}\,a\sqrt{a\pi}.$$

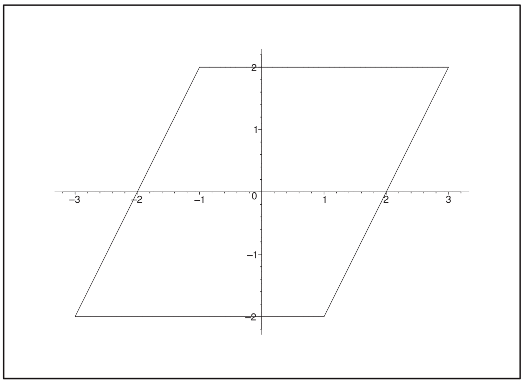

Figure 50: The domain $B(T)$ for $T = 2$ and $p = \dfrac{1}{2}$

3) Since the integrand is positive we do not have to consider the parallelogram $B(T)$. It is actually possible directly to use the change of variables

$$(x, y) = (py + t, y).$$

Nevertheless, we shall here stick to the analysis of $B(T)$.

The parallelogram is described by $-T + py \le x \le T + py$, $y \in [-T, T]$. If we put $x = py + t$, we get the conditions

$$t \in [-T, T] \text{ and } y \in [-T, T], \qquad \text{i.e. } (t, y) \in [-T, T] \times [-T, T].$$

Furthermore,

$$\begin{aligned} x^2 - 2pxy + y^2 &= (x^2 - 2pxy + p^2y^2) + (1 - p^2)y^2 \\ &= (x - py)^2 + (1 - p^2)y^2 = t^2 + (1 - p^2)y^2. \end{aligned}$$

Finally, the corresponding Jacobian is given by

$$\begin{vmatrix} \dfrac{\partial x}{\partial t} & \dfrac{\partial x}{\partial y} \\[2ex] \dfrac{\partial y}{\partial t} & \dfrac{\partial y}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & p \\ 0 & 1 \end{vmatrix} = 1 \cdot 1 - p \cdot 0 = 1,$$

thus

$$\begin{aligned} \int_{B(T)} \exp\left(-\frac{x^2 - 2pxy + y^2}{2(1 - p^2)}\right) dS &= \int_{[-T,T]\times[-T,T]} \exp\left(-\frac{t^2}{2(1 - p^2)} - \frac{y^2}{2}\right) dt\, dy \\ &= \int_{-T}^{T} \exp\left(-\frac{t^2}{2(1 - p^2)}\right) dt \cdot \int_{-T}^{T} \exp\left(-\frac{y^2}{2}\right) dy. \end{aligned}$$

It follows from 1) by taking the limit $T \to +\infty$ and choosing $a = 2(1-p^2)$ and $a = 2$ respectively that

$$\int_{\mathbb{R}^2} \exp\left(-\frac{x^2 - 2pxy + y^2}{2(1 - p^2)}\right) dS = \sqrt{2(1 - p^2)\pi} \cdot \sqrt{2\pi} = 2\pi\sqrt{1 - p^2}.$$

4) Now $p \in]-1,1[$, so

$$x^2-2pxy+y^2=(x-py)^2+(1-p^2)y^2=(1-p^2)x^2+(y-px)^2,$$

and it then follows by the rules of magnitudes that the integral is convergent.

Finally, by the method of 3),

$$\begin{aligned}\int_{B(T)} xy\,\exp\left(-\frac{x^2-2pxy+y^2}{2(1-p^2)}\right)dS &= \int_{[-T,T]^2}(py+t)y\,\exp\left(-\frac{t^2}{2(1-p^2)}-\frac{y^2}{2}\right)dt\,dy\\ &= p\int_{-T}^{T}\exp\left(-\frac{t^2}{2(1-p^2)}\right)dt\cdot\int_{-T}^{T}y^2\exp\left(-\frac{y^2}{2}\right)dy+0,\end{aligned}$$

because e.g. $y\,\exp\left(-\dfrac{y^2}{2}\right)$ is odd. Then by the limit $T\to+\infty$ and an application of the previous results we end up with

$$\begin{aligned}\int_{\mathbb{R}^2} xy\,\exp\left(-\frac{x^2-2pxy+y^2}{2(1-p^2)}\right)dS &= p\int_{-\infty}^{+\infty}\exp\left(-\frac{t^2}{2(1-p^2)}\right)dt\cdot\int_{-\infty}^{+\infty}y^2\exp\left(-\frac{y^2}{2}\right)dy\\ &= p\sqrt{2\pi(1-p^2)}\cdot\frac{2}{2}\sqrt{2\pi}=2\pi p\sqrt{1-p^2}.\end{aligned}$$

Example 6.8 *Check if the improper space integral*

$$\int_{\mathbb{R}^3}\frac{z}{c^2+x^2+y^2+z^2}\,d\Omega$$

is convergent or divergent. In case of convergency, find its value.

A Improper space integral.

D Consider the space integral in the domain where the integrand is positive.

We see by inspection that the degree of the denominator is only 1 bigger than the degree of the numerator. Therefore, a qualified guess is of course that the integral is divergent.

I Let us prove this in the traditional way. First note that the integrand is ≥ 0 for $z\geq 0$. Let $\Omega(R)$ denote the half ball in the upper half space $z\geq 0$ of centrum $\mathbf{0}$ and radius R. Then we get in spherical coordinates that

$$\begin{aligned}\int_{\Omega(R)}\frac{z}{c^2+x^2+y^2+z^2}\,d\Omega &= \int_0^{\frac{\pi}{2}}\left\{\int_0^{2\pi}\left(\int_0^R\frac{r\cos\theta}{c^2+r^2}\cdot r^2\sin\theta\,dr\right)d\varphi\right\}d\theta\\ &= 2\pi\left[\frac{\sin^2\theta}{2}\right]_0^{\frac{\pi}{2}}\int_0^R\frac{r^3}{c^2+r^2}\,dr=\pi\int_0^R\left(r-c^2\cdot\frac{r}{c^2+r^2}\right)dr\\ &= \pi\left[\frac{1}{2}r^2-\frac{c^2}{2}\ln\left(c^2+r^2\right)\right]_0^R=\frac{\pi}{2}\left\{R^2-c^2\ln(c^2+R^2)+2c^2\ln c\right\}\\ &\to+\infty \qquad\text{for } R\to+\infty,\end{aligned}$$

and the improper space integral is divergent as claimed above.

7 Transformation of space integrals

Example 7.1 *Let A denote the tetrahedron, which is bounded by the four planes of the equations*

$$x+y=1, \quad y+z=1, \quad z+x=1, \quad x+y+z=1.$$

Compute the space integral

$$I=\int_A (x+y)(y+z)\,dx\,dy\,dz$$

by introducing the new variables

$$u=1-x-y, \quad v=1-y-z, \quad w=1-z-x.$$

A Transformation of a space integral.

Find the Jacobian and the limits of u, v and w.

I We derive from

$$dx\,dy\,dz = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw = \left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right|^{-1} du\,dv\,dw$$

and

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ -1 & 0 & -1 \end{vmatrix} = -1-1+0-0-0-0 = -2$$

that the weight function is

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right|^{-1} = \frac{1}{2}.$$

The integrand is

$$(x+y)(y+z) = (1-u)(1-v).$$

Considering the limits of u, v and w we see that

$$\begin{array}{rcl} x+y=1 & \text{corresponds to} & u = 1-x-y = 0, \\ y+z=1 & \text{corresponds to} & v = 1-y-z = 0, \\ z+x=1 & \text{corresponds to} & w = 1-z-x = 0. \end{array}$$

From

$$u+v+w = 3-2(x+y+z),$$

follows that

$$x+y+z=1 \quad \text{corresponds to} \quad u+v+w=1.$$

Finally, the tetrahedron lies in the first octant of the XYZ-space, where $x+y \leq 1$, $y+z \leq 1$ and $z+x \leq 1$. Hence the domain in the UVW-space is

$$\begin{aligned} B &= \{(u,v,w) \mid u \geq 0,\, v \geq 0,\, w \geq 0,\, u+v+w \leq 1\} \\ &= \{(u,v,w) \mid 0 \leq u \leq 1,\, 0 \leq v \leq 1-u,\, 0 \leq w \leq 1-u-v\}. \end{aligned}$$

By this transformation followed by a reduction in rectangular coordinates we get

$$
\begin{aligned}
I &= \int_A (x+y)(y+z)\,dx\,dy\,dz = \int_B (1-u)(1-v)\cdot\frac{1}{2}\,du\,dv\,dw \\
&= \frac{1}{2}\int_0^1 (1-u)\left\{\int_0^{1-u}(1-v)\left\{\int_0^{1-u-v} dw\right\}dv\right\}du \\
&= \frac{1}{2}\int_0^1 (1-u)\left\{\int_0^{1-u}(1-v)(1-u-v)dv\right\}du \\
&= \frac{1}{2}\int_0^1 (1-u)\left\{\int_0^{1-u}\left\{(v-1)^2+u(v-1)\right\}dv\right\}du \\
&= \frac{1}{2}\int_0^1 (1-u)\left[\frac{1}{3}(v-1)^3+\frac{u}{2}(v-1)^2\right]_0^{1-u} du \\
&= \frac{1}{2}\int_0^1 (1-u)\left\{\frac{1}{3}-\frac{1}{3}u^2 3+\frac{u}{2}\cdot u^2-\frac{u}{2}\right\}du \\
&= \frac{1}{2}\int_0^1 (1-u)\left\{\frac{1}{3}+\frac{1}{6}u^3-\frac{u}{2}\right\}du = \frac{1}{12}\int_0^1 (1-u)(2+u^3-3u)\,du \\
&= \frac{1}{12}\int_0^1 \left\{2+u^3-3u-2u-u^4+3u^2\right\}du = \frac{1}{12}\left\{2+\frac{1}{4}-\frac{3}{2}-\frac{2}{2}-\frac{1}{5}+\frac{3}{3}\right\} \\
&= \frac{1}{12}\cdot\frac{1}{30}(60+15-45-6) = \frac{1}{12}\cdot\frac{1}{30}\cdot 24 = \frac{1}{15}.
\end{aligned}
$$

Example 7.2 *Let A be the closed point set in $\mathbb{R}^3$, which is bounded by the four elliptic paraboloids of the equations*

$$(1) \quad z = \frac{3}{2} - \frac{1}{6}\,x^2 - \frac{2}{3}\,y^2$$

$$(2) \quad z = \frac{1}{2} - \frac{1}{2}\,x^2 - 2y^2,$$

$$(3) \quad z = -1 + \frac{1}{4}\,x^2 + y^2,$$

$$(4) \quad z = -2 + \frac{1}{8}\,x^2 + \frac{1}{2}\,y^2.$$

The point set A intersects the ZX-plane in a point set B_1, and the YZ-plane in a point set B_2.

1) Sketch B_1 and B_2.

2) Compute the volume $Vol(A)$ and the space integral

$$I = \int_A \frac{1}{\sqrt{x^2+4y^2+z^2}}\,dx\,dy\,dz$$

by introducing the new variables (u,v,w), such that

$$x = \sqrt{uv}\,\cos w, \quad y = \frac{1}{2}\,\sqrt{uv}\,\sin w, \quad z = \frac{1}{2}\,(u-v),$$

where

$$u,\, v \in [0,+\infty[, \qquad w \in [0,2\pi].$$

A Transformation of a space integral.

D First sketch B_1 (put $y=0$) and B_2 (put $x=0$). Then apply the transformation formula, i.e. compute the weight function and change variables.

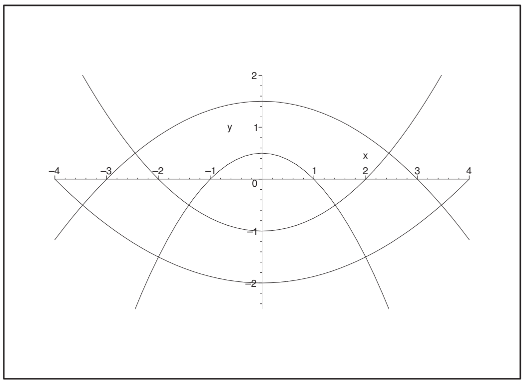

Figure 51: The set B_1 is the union of the two "skew" quadrilateral sets.

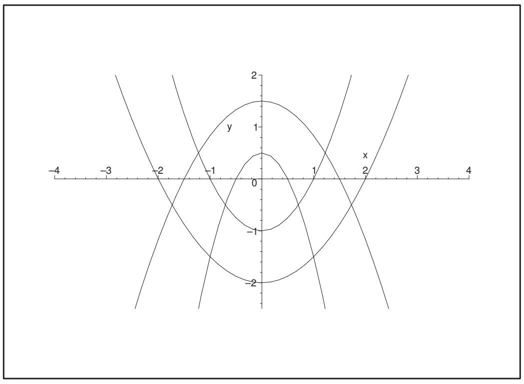

Figure 52: The set B_2 is the union of the two "skew" quadrilateral sets.

I 1) By putting $y=0$, we get in the XZ-plane the four parabolas

$$z=\frac{3}{2}-\frac{1}{6}\,x^2,\quad z=\frac{1}{2}-\frac{1}{2}\,x^2,\quad z=-1+\frac{1}{4}\,x^2,\quad z=-2+\frac{1}{8}\,x^2,$$

and it is easy to sketch B_1.

By putting $x=0$, we get in the YZ-plane

$$z=\frac{3}{2}-\frac{2}{3}\,y^2,\quad z=\frac{1}{2}-2y^2,\quad z=-1+y^2,\quad z=-2+\frac{1}{2}\,y^2,$$

and it is easy to sketch B_2.

2) Let

$$x=\sqrt{uv}\,\cos w,\quad y=\frac{1}{2}\,\sqrt{uv}\,\sin w,\quad z=\frac{1}{2}\,(u-v),$$

where u, $v\geq 0$ and $w\in[0,2\pi]$. We shall first find the image of A by this transformation.

a) By insertion into the boundary surface

$$z=\frac{3}{2}-\frac{1}{6}\,x^2-\frac{2}{3}\,y^2$$

we get

$$\frac{1}{2}\,(u-v)=\frac{3}{2}-\frac{1}{6}\,uv\cos^2 w-\frac{1}{6}\,uv\sin^2 w=\frac{3}{2}-\frac{1}{6}\,uv,$$

thus $3(u-v)=9-uv$, which is reformulated as

$$uv+3u=u(v+3)=9+3v=3(v+3).$$

It follows from $v\geq 0$ that $u=3$, hence this boundary surface is mapped into a part of the plane $u=3$.

b) By insertion into the boundary surface

$$z = \frac{1}{2} - \frac{1}{2}\,x^2 - 2y^2$$

we get

$$\frac{1}{2}\,(u-v) = \frac{1}{2} - \frac{1}{2}\,uv\cos^2 w - \frac{1}{2}\,uv\sin^2 w = \frac{1}{2} - \frac{1}{2}\,uv,$$

i.e. $u - v = 1 - uv$, and thus

$$uv + u = u(v+1) = v+1.$$

From $v \geq 0$ follows that $u = 1$, hence the boundary surface is mapped into a part of the plane $u = 1$.

c) We get by insertion into the boundary surface

$$z = -1 + \frac{1}{4}\,x^2 + y^2$$

that

$$\frac{1}{2}\,(u-v) = -1 + \frac{1}{4}\,uv\cos^2 w + \frac{1}{4}\,uv\sin^2 w = -1 + \frac{1}{4}\,uv,$$

thus $2(u-2) = uv - 4$, and hence

$$uv + 2v = v(u+2) = 2u + 4 = 2(u+2).$$

It follows from $u \geq 0$ that $v = 2$, so the boundary surface is mapped into a part of the plane $v = 2$.

d) We get by insertion into the boundary surface

$$z = -2 + \frac{1}{8}\,x^2 + \frac{1}{2}\,y^2$$

that

$$\frac{1}{2}\,(u-v) = -2 + \frac{1}{8}\,uv\cos^2 w + \frac{1}{8}\,uv\sin^2 w = -2 + \frac{1}{8}\,uv,$$

thus $4(u-v) = -16 + uv$, and hence

$$uv + 4v = v(u+4) = 4u + 16 = 4(u+4).$$

It follows from $u \geq 0$ that $v = 4$, so the boundary surface is mapped into a part of the plane $v = 4$.

The only condition on w is that $(\cos w, \sin w)$ shall encircle the unit circle only once, so $w \in [0, 2\pi]$.

By the transformation A is mapped onto the set

$$B = [1,3] \times [2,4] \times [0, 2\pi].$$

Then we calculate the Jacobian (for $u,\, v>0$)

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(u,v,w)} &= \left|\begin{array}{ccc} \frac{1}{2}\sqrt{\frac{v}{u}}\,\cos w & \frac{1}{2}\sqrt{\frac{u}{v}}\,\cos w & -\sqrt{uv}\,\sin w \\ \frac{1}{4}\sqrt{\frac{v}{u}}\,\sin w & \frac{1}{4}\sqrt{\frac{u}{v}}\,\sin w & \frac{1}{2}\sqrt{uv}\,\cos w \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right| \\
&= \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{2}\sqrt{uv}\left|\begin{array}{ccc} \sqrt{\frac{v}{u}}\,\cos w & \sqrt{\frac{u}{v}}\,\cos w & -2\sin w \\ \sqrt{\frac{v}{u}}\,\sin w & \sqrt{\frac{u}{v}}\,\sin w & 2\cos w \\ 1 & -1 & 0 \end{array}\right| \\
&= \frac{1}{16}\sqrt{uv}\cdot 2\left|\begin{array}{ccc} \sqrt{\frac{v}{u}}\,\cos w & \left(\sqrt{\frac{u}{v}}+\sqrt{\frac{v}{u}}\right)\cos w & -\sin w \\ \sqrt{\frac{v}{u}}\,\sin w & \left(\sqrt{\frac{u}{v}}+\sqrt{\frac{v}{u}}\right)\sin w & \cos w \\ 1 & 0 & 0 \end{array}\right| \\
&= \frac{1}{8}\sqrt{uv}\left(\sqrt{\frac{u}{v}}+\sqrt{\frac{v}{u}}\right)\left|\begin{array}{cc} \cos w & -\sin w \\ \sin w & \cos w \end{array}\right| = \frac{1}{8}\,(u+v)>0.
\end{aligned}$$

We get by the transformation formula,

$$\begin{aligned}
\text{Vol}(A) &= \int_A dx\,dy\,dz = \int_B \frac{\partial(x,y,z)}{\partial(u,v,w)}\,du\,dv\,dw = \int_0^{2\pi}\left\{\int_2^4\left\{\int_1^3 \frac{1}{8}\,(u+v)\,du\right\}dv\right\}dw \\
&= \frac{1}{8}\cdot 2\pi\int_2^4\left[\frac{u^2}{2}+uv\right]_{u=1}^3 dv = \frac{\pi}{8}\int_2^4\{9+6v-1-2v\}dv \\
&= \frac{\pi}{8}\int_2^4\{4v+8\}dv = \frac{\pi}{8}\left[2v^2+8v\right]_2^4 = \frac{\pi}{4}\,\left[v^2+4v\right]_2^4 \\
&= \frac{\pi}{4}\{16+16-4-8\} = \pi\{4+4-1-2\} = 5\pi.
\end{aligned}$$

Let us turn to the space integral. Since

$$x^2+4y^2+z^2 = uv\cos^2 w + uv\sin^2 w + \frac{1}{4}\,(u-v)^2 = \frac{1}{4}\left\{(u-v)^2+4uv\right\} = \frac{1}{4}\,(u+v)^2,$$

and $u+v>0$, the integrand is transformed into

$$\frac{1}{\sqrt{x^2+4y^2+z^2}} = \frac{2}{u+v}.$$

Finally, by the transformation formula,

$$\begin{aligned}
&\int_A \frac{1}{\sqrt{x^2+4y^2+z^2}}\,dx\,dy\,dz = \int_B \frac{1}{u+v}\cdot\frac{\partial(x,y,z)}{\partial(u,v,w)}\,du\,dv\,dw \\
&\quad= \int_B \frac{2}{u+v}\cdot\frac{u+v}{8}\,du\,dv\,dw = \frac{1}{4}\int_B du\,dv\,dw = \frac{1}{4}\,\text{Vol}(B) \\
&\quad= \frac{1}{4}\cdot 2\cdot 2\cdot 2\pi = 2\pi.
\end{aligned}$$

Example 7.3 *We can write the formula of transformation of a space integral in the following way,*

$$\int_{\tilde{\Omega}} f(\tilde{\mathbf{x}})\, d\tilde{\Omega} = \int_{\Omega} f(\tilde{\mathbf{x}}(\mathbf{x}))\, |J(\mathbf{x})|\, d\Omega,$$

where J is the determinant of that matrix, the elements of which are

$$\frac{\partial \tilde{\mathbf{x}}_i}{\partial x_j}, \qquad i,\, j \in \{1,2,3\}.$$

We shall in particular interpret $\tilde{\Omega}$ as created from Ω by a translation *and a* deformation*. This means that to the point $\mathbf{x}$ we let correspond a point $\tilde{\mathbf{x}}$ given by*

$$\tilde{\mathbf{x}} = \mathbf{x} + \mathbf{u}(\mathbf{x}),$$

where $\mathbf{u}$ is the displacement vector field*. When $\mathbf{u}$ is a constant vector, we get a translation. However, in general $\mathbf{u}$ varies in space (as indicated by the notation), such that we have a combination of a translation and a deformation.*

1. *Compute $J(x,y,z)$ by introducing $\mathbf{u} = (u_x, u_y, u_z)$.*

In the Theory of Elasticity the deformations are often small *in the sense that the derivative of $\mathbf{u}$ is small, so we can reject all there* products*.*

2. *Prove that by this assumption, $J = 1 + \; div\, \mathbf{u}$.*

3. *Finally, prove that the divergence is the relative increase of the volume corresponding to the deformation.*

A Transformation of space integrals.

D Calculate the first approximation of the Jacobian.

I 1) The transformation is given by

$$\begin{aligned} \tilde{x}_1 &= x + u_x(\mathbf{x}), \\ \tilde{x}_2 &= y + u_y(\mathbf{x}), \qquad \mathbf{x} = (x,y,z), \\ \tilde{x}_3 &= z + u_z(\mathbf{x}), \end{aligned}$$

hence

$$\begin{aligned} J(x,y,z) &= \begin{vmatrix} 1 + \dfrac{\partial u_x}{\partial x} & \dfrac{\partial u_y}{\partial x} & \dfrac{\partial u_z}{\partial x} \\[2ex] \dfrac{\partial u_x}{\partial y} & 1 + \dfrac{\partial u_y}{\partial y} & \dfrac{\partial u_z}{\partial y} \\[2ex] \dfrac{\partial u_x}{\partial z} & \dfrac{\partial u_y}{\partial z} & 1 + \dfrac{\partial u_z}{\partial z} \end{vmatrix} \\ &= 1 + \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} + \text{ products of higher order.} \end{aligned}$$

2) If we remove all products of higher order, then we get

$$J = 1 + \text{ div } \mathbf{u}.$$

3) The geometrical interpretation of J is given by

$$d\tilde{\Omega} = |J(\mathbf{x})|\, d\Omega,$$

where $d\tilde{\Omega}$ and $d\Omega$ are considered as infinitesimal volumes corresponding to each other. This may possibly be clarified by

$$\Delta\tilde{\Omega} \approx |J(\mathbf{x})|\, \Delta\Omega.$$

Assume that div $\mathbf{u}$ is small, so higher order terms can be rejected. Then

$$J = 1 + \text{ div } \mathbf{u} > 0,$$

and thus

$$d\tilde{\Omega} = \{1 + \text{ div } \mathbf{u}\} d\Omega.$$

The factor $1 + \text{ div } \mathbf{u}$ indicates the quotient between the two infinitesimal volumes, so div $\mathbf{u}$ can be interpreted as the relative signed increase of the volume.

Example 7.4 *Let $A \subset \mathbb{R}^3$ be given by*

$$0 \le x, \quad 0 \le y, \quad 0 \le z, \quad \sqrt{x} + \sqrt{y} + \sqrt{z} \le 1.$$

Compute the volume of A and the space integral

$$I = \int_A \exp\left[\left(\sqrt{x} + \sqrt{y} + \sqrt{z}\right)^6\right] dx\, dy\, dz$$

by introducing the new variables

$$u = \sqrt{x} + \sqrt{y}, \quad v = \sqrt{x} - \sqrt{y}, \quad w = \sqrt{x} + \sqrt{y} + \sqrt{z}.$$

A Transformation of space integrals.

D Find the inverse transformation and compute the Jacobian before the transformation formula is applied.

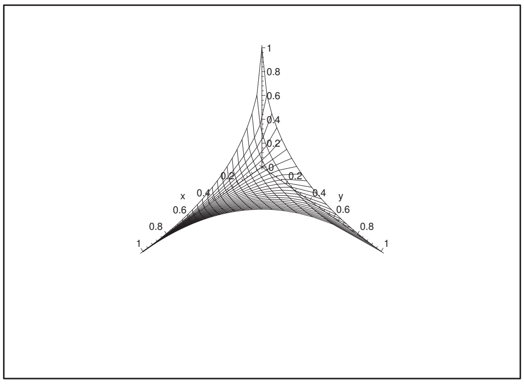

Figure 53: The domain A.

I We derive from

$$2\sqrt{x} = u + v, \quad 2\sqrt{y} = u - v, \quad \sqrt{z} = w - u,$$

that

$$x = \frac{1}{4}(u+v)^2, \quad y = \frac{1}{4}(u-v)^2, \quad z = (w-u)^2.$$

Then find the parametric domain B in the (u, v, w)-space.

1) The boundary surface $x = 0$ is mapped into the plane $v = -u$.
2) The boundary surface $y = 0$ is mapped into the plane $v = u$.
3) The boundary surface $z = 0$ is mapped into the plane $w = u$.
4) The boundary surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$ is mapped into the plane $w = 1$.

The set A is closed and bounded, and the transformation is continuous. It therefore follows from the second main theorem for continuous functions that A is transformed into the closed and bounded parametric domain

$$B = \{(u, v, w) \mid 0 \le w \le 1,\, 0 \le u \le w,\, -u \le v \le u\}.$$

Then the Jacobian is given by

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(u,v,w)} &= \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\[2mm] \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{1}{2}(u+v) & \frac{1}{2}(u+v) & 0 \\[2mm] \frac{1}{2}(u-v) & -\frac{1}{2}(u-v) & 0 \\[2mm] -2(w-u) & 0 & 2(w-u) \end{vmatrix} \\
&= \frac{1}{2}(u+v)\cdot\frac{1}{2}(u-v)\cdot 2(w-u) \begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{vmatrix} \\
&= \frac{1}{2}\left(u^2-v^2\right)(w-u)\cdot(-2) = -\left(u^2-v^2\right)(w-u).
\end{aligned}$$

Since $v^2 \le u^2$ and $u \le w$ in B, it follows from the transformation formula that

$$\begin{aligned}
\mathrm{vol}(A) &= \int_A d\Omega = \int_V \left|-\left(-u^2-v^2\right)(w-u)\right| \, du\, dv\, dw \\
&= \int_0^1 \left\{ \int_0^w \left[\int_{-u}^u \left(u^2-v^2\right)(w-u)\, dv \right] du \right\} dw \\
&= \int_0^1 \left\{ \int_0^w (w-u)\cdot\frac{4}{3}u^3\, du \right\} dw = \int_0^1 \left\{ \frac{4}{3}\int_0^w \left(wu^3-u^4\right) du \right\} dw \\
&= \int_0^1 \frac{4}{3}\left[\frac{1}{4}wu^4 - \frac{1}{5}u^5\right]_{u=0}^{w} dw = \int_0^1 \frac{1}{15}w^5\, dw = \frac{1}{90},
\end{aligned}$$

and

$$\begin{aligned}
I &= \int_A \exp\left[\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^6\right] d\Omega \\
&= \int_0^1 \exp\left(w^6\right) \left\{ \int_0^w \left[\int_{-u}^u \left(u^2-v^2\right)(w-u)\, dv \right] du \right\} dw \\
&= \int_0^1 \frac{1}{15}\exp\left(w^6\right)\cdot w^5\, dw = \frac{1}{90}\int_0^1 e^t\, dt = \frac{e-1}{90},
\end{aligned}$$

where we also found that

$$\int_0^w \left\{ \int_{-u}^u \left(u^2 - v^2\right)(w-u)\, dv \right\} du = \frac{1}{15}\, w^5.$$

Example 7.5 *Let B be the triangle given by $x \geq 0$, $y \geq 0$, $x + y \leq 1$. Compute the improper plane integral*

$$I = \int_B \exp\left(\frac{x-y}{x+y}\right) dS$$

by introducing the new variables $(u, v) = (x + y, x - y)$.

A Transformation of an improper plane integral.

D The integrand is not defined at $(x, y) = (0, 0) \in B$. Otherwise, the integrand is positive, so in the worst case we shall only get that the value becomes $+\infty$.

Find x and y expressed by u and v. Find the parametric domain in the (u, v)-plane. Compute the Jacobian. Finally, insert into the transformation formula, check if the singularity has any effect and compute.

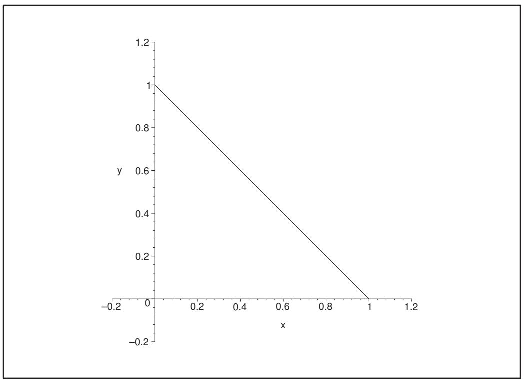

Figure 54: The domain of integration B.

I The transformation is continuous with a continuous inverse:

$$u = x + y \qquad \text{og} \qquad v = x - y,$$
$$x = \frac{1}{2}(u+v) \qquad \text{og} \qquad y = \frac{1}{2}(u-v).$$

Furthermore, B is closed and bounded, so by using the second main theorem for continuous functions we conclude that the image, i.e. the new parametric domain A in the (u,v)-plane is also closed and bounded. It therefore suffices to find the images of the boundary curves.

1) $x = 0$ is mapped into $u + v = 0$, i.e. into $v = -u$.

2) $y = 0$ is mapped into $u - v = 0$, i.e. into $v = u$.

3) $x + y = 1$ is mapped into $u = 1$.

It follows that A in the (u,v)-plane is the triangle which is defined by these three lines, hence

$$A = \{(u,v) \mid 0 \le u \le 1,\ -u \le v \le u\}.$$

Then compute the Jacobian,

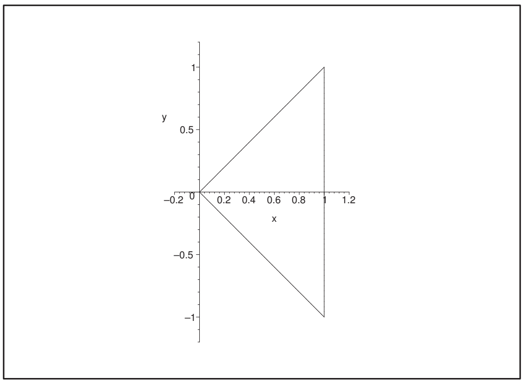

Figure 55: The parametric domain A in the (u,v)-plane.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial u}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}.$$

Finally, by putting into the transformation formula where we also have in mind that the integral is improper of a positive integrand:

$$\begin{aligned} I &= \int_B \exp\left(\frac{x-y}{x+y}\right) dS = \int_A \exp\left(\frac{v}{u}\right) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv \\ &= \frac{1}{2}\int_0^1 \left\{\int_{-u}^{u} \exp\left(\frac{v}{u}\right) dv\right\} du = \frac{1}{2}\lim_{\varepsilon\to 0+}\int_\varepsilon^1 \left\{\int_{-u}^{u} \exp\left(\frac{v}{u}\right) dv\right\} du \\ &= \frac{1}{2}\lim_{\varepsilon\to 0+}\int_\varepsilon^1 u\left[\exp\left(\frac{v}{u}\right)\right]_{v=-u}^{u} du = \frac{1}{2}\lim_{\varepsilon\to 0+}\int_\varepsilon^1 u\left(e-e^{-1}\right) du \\ &= \sinh 1\int_0^1 u\, du = \frac{\sinh 1}{2}. \end{aligned}$$

Example 7.6 *Let A be the tetrahedron which is bounded by the four planes of the equations*

$$x+y+z=0, \quad x+y-z=0, \quad x-y-z=0, \quad 2x-z=1.$$

Compute the space integral

$$I = \int_A (x+y+z)(x+y-z)(x-y-z)\, dx\, dy\, dz$$

by introducing the new variables

$$u = x+y+z, \quad v = x+y-z, \quad w = x-y-z.$$

A Transformation of a space integral.

D Find x, y, z expressed by u, v, w. Then find the parametric domain B in the (u,v,w)-space which is uniquely mapped onto A. Compute the Jacobian, and finally, apply the transformation formula.

I It follows from

$$u = x+y+z, \quad v = x+y-z, \quad w = x-y-z,$$

that

$$u+w = 2x, \qquad \text{i.e.} \quad x = \frac{1}{2}(u+w).$$

Then

$$u-v = 2z, \qquad \text{i.e.} \quad z = \frac{1}{2}(u-v)$$

and

$$v-w = 2y, \qquad \text{i.e.} \quad y = \frac{1}{2}(v-w).$$

Summarizing,

$$u = x + y + z, \qquad v = x + y - z, \qquad w = x - y - z,$$
$$x = \frac{1}{2}(u + w), \qquad y = \frac{1}{2}(v - w), \qquad z = \frac{1}{2}(u - v),$$

and the Jacobian is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{vmatrix} = \frac{1}{8}\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} = \frac{1}{8}(-1-1) = -\frac{1}{4}.$$

Now we shall find the images of the boundary surfaces of the tetrahedron:

1) $x + y + z = 0$ is mapped into $u = 0$.
2) $x + y - z = 0$ is mapped into $v = 0$.
3) $x - y - z = 0$ is mapped into $w = 0$.
4) $2x - z = 1$, i.e. $2 = 4x - 2z$, is mapped into

$$2 = 2u + 2w - u + v = u + v + 2w, \quad \text{i.e.} \quad u + v + 2w = 2.$$

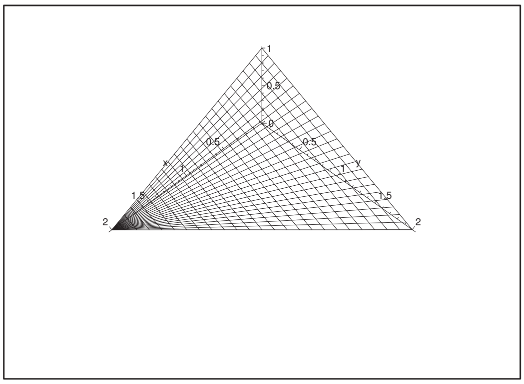

Figure 56: The transformed parametric domain B.

The inverse transformation is continuous, and A is closed and bounded. Hence, A is transformed into a new tetrahedron B as indicated on the figure. Notice that B is cut at the height $w \in [0,1[$ in the triangle

$$B(w) = \{(u,v) \mid u \geq 0,\, v \geq 0,\, u + v \leq 2(1-w)\}.$$

This can be exploited in the calculation of the transformed integral by the method of slicing.

According to the transformation theorem,

$$\begin{aligned}
I &= \int_A (x+y+z)(x+y-z)(x-y-z)\,dx\,dy\,dz \\
&= \int_B uvw \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw = \frac{1}{4}\int_0^1 w\left\{\int_{B(w)} uv\,dS\right\} dw.
\end{aligned}$$

Then compute the integral over the slice at height w,

$$\begin{aligned}
\int_{B(w)} uv\,dS &= \int_0^{2(1-w)} u\left\{\int_0^{2(1-w)-u} v\,dv\right\} du = \int_0^{2(1-w)} u\cdot\frac{1}{2}\{2(1-w)-u\}^2\,du \\
&= \frac{1}{2}\int_0^{2(1-w)} \left\{4(1-w)^2u - 4(1-w)u^2 + u^3\right\} dy \\
&= \frac{1}{2}\left[2(1-w)^2u^2 - \frac{4}{3}(1-w)u^3 + \frac{1}{4}u^4\right]_0^{2(1-w)} \\
&= \frac{1}{2}\left\{2(1-w)^2\cdot 4(1-w)^2 - \frac{4}{3}(1-w)\cdot 8(1-w)^3 + \frac{1}{4}\cdot 16(1-w)^4\right\} \\
&= \frac{1}{2}(1-w)^4\left\{8 - \frac{32}{3} + 4\right\} = \frac{1}{2}\cdot\frac{4}{3}(1-w)^4 = \frac{2}{3}(w-1)^4.
\end{aligned}$$

Finally, by insertion,

$$\begin{aligned} I &= \frac{1}{4}\int_0^1 w\left\{\int_{B(w)} uv\,dS\right\} dw = \frac{1}{4}\cdot\frac{2}{3}\int_0^1 w(w-1)^4\,dw \\ &= \int 16\int_0^1 \left\{(w-1)^5+(w-1)^4\right\} dw = \frac{1}{6}\left[\frac{1}{6}(w-1)^6+\frac{1}{5}(w-1)^5\right]_0^1 \\ &= \frac{1}{6}\left\{-\frac{1}{6}-\frac{1}{5}(-1)\right\} = \frac{1}{6}\left(\frac{1}{5}-\frac{1}{6}\right) = \frac{1}{180}. \end{aligned}$$

Example 7.7 *Let K denote the closed ball of centrum $(1,1,1)$ and radius $\sqrt{3}$. We construct a subset $A \subset K$ by only keeping those points from K in A, which furthermore satisfy $r \geq 1$ and lie in the first octant.*
Compute the space integral

$$I = \int_A \frac{1}{r^6}\,d\Omega$$

by introducing the new variables

$$u = \frac{x}{r^2}, \qquad v = \frac{y}{r^2}, \qquad w = \frac{z}{r^2}.$$

A Transformation of a space integral. This is the ‘simplest” non-trivial example in the three dimensional space. We shall see that even in this case the computations grow very big.

D First find A, and then the parametric domain D of the variables (u, v, w). Compute the Jacobian, and finally also the transformed integral.

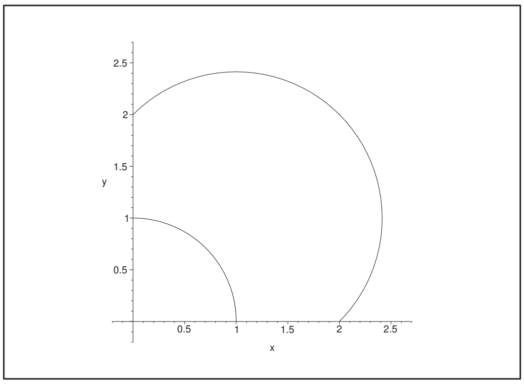

Figure 57: The boundary surface of A in each of the three planes $x = 0$, $y = 0$, or $z = 0$.

I The set A is described by

$$A = \left\{(x,y,z) \mid (x-1)^2+(y-)^2+(z-1)^2 \leq 3, x^2+y^2+z^2 \geq 1, x \geq 0, y \geq 0, z \geq 0\right\}.$$

The boundary surface in each of the planes $x = 0$, $y = 0$ and $z = 0$ is indicated on the figure.

Then check the image in the (u, v, w)- space of each of the boundary surfaces in the (x, y, z)-space.

Clearly, the boundary surface $x^2 + y^2 + z^2 = 1$ is mapped into $u^2 + v^2 + z^2 = 1$, and they both lie in the first octant.

Then check the transformation of the boundary surface

$$(x - 1)^2 + (y - 1)^2 + (z - 1)^2 = 3.$$

If we put

$$R^2 = u^2 + v^2 + w^2 = \frac{x^2}{r^4} + \frac{y^2}{r^4} + \frac{z^2}{r^4} = \frac{1}{r^2},$$

then

$$x = u \cdot r^2 = \frac{u}{R^2}, \qquad y = \frac{v}{R^2}, \qquad z = \frac{w}{R^2},$$

hence by insertion

$$(u - R^2)^2 + (c - R^2)^2 + (w - R^2)^2 = 3R^4,$$

and thus by a computation

$$\begin{aligned} 3R^4 &= u^2 - 2uR^2 + R^4 + v^2 - 2vR^2 + w^2 - 2wR^2 + R^4 \\ &= (u^2 + v^2 + w^2) - 2(u + v + w)R^2 + 3R^4 \\ &= R^2 - 2(u + v + w)R^2 + 3R^4 = 3R^4 + R^2\{1 - 2(u + v + w)\}. \end{aligned}$$

Now, $R^2 = u^2 + v^2 + w^2 = \dfrac{1}{r^2} > 0$, to this is reduced to the equation of a plane surface in the first octant,

$$u + v + w = \frac{1}{2}.$$

We conclude that D is that part of the closed first octant, which also lies between the plane $u + v + w = \dfrac{1}{2}$ and the sphere $u^2 + v^2 + w^2 = 1$.

Since we have

$$u + v + w = R\{\sin\theta(\cos\varphi + \sin\varphi) + \cos\theta\}$$

in spherical coordinates

$$u = R\,\sin\theta\cos\varphi, \quad v = R\,\sin\theta\sin\varphi, \quad w = R\cos\theta,$$

we get the following description of D in spherical coordinates

$$D = \left\{(R, \varphi, \theta) \mid [2\{\sin\theta(\cos\varphi + \sin\varphi) + \cos\theta\}]^{-1} \le R \le 1, 0 \le \varphi, \theta \le \frac{\pi}{2}\right\}.$$

Then calculate the Jacobian $\dfrac{\partial(x, y, x)}{\partial(u, v, w)}$, where we use that

$$(x, y, z) = \left(\frac{u}{R^2}, \frac{v}{R^2}, \frac{w}{R^2}\right).$$

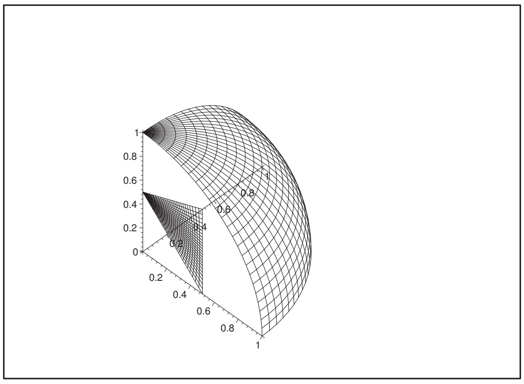

Figure 58: The domain D lies in the first octant between the two surfaces.

First note that e.g.

$$\frac{\partial}{\partial u}\left(\frac{1}{R^2}\right) = -\frac{1}{R^4}\cdot\frac{\partial R^2}{\partial u} = -\frac{2u}{R^4},$$

and similarly of symmetric reasons,

$$\frac{\partial}{\partial v}\left(\frac{1}{R^2}\right) = -\frac{2v}{R^4} \quad \text{og} \quad \frac{\partial}{\partial w}\left(\frac{1}{R^2}\right) = -\frac{2w}{R^4},$$

Then

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(u,v,w)} &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{1}{R^2}-\frac{2u^2}{R^4} & -\frac{2uv}{R^4} & -\frac{2uw}{R^4} \\ -\frac{2uv}{R^4} & \frac{1}{r^2}-\frac{2v^2}{R^4} & -\frac{2vw}{R^4} \\ -\frac{2uw}{R^4} & -\frac{2vw}{R^4} & \frac{1}{R^2}-\frac{2w^2}{R^4} \end{vmatrix} \\
&= \frac{1}{R^{12}} \begin{vmatrix} -u^2+v^2+w^2 & -2uv & -2uw \\ -2uv & u^2-v^2+w^2 & -2vw \\ -2uw & -2vw & u^2+v^2-w^2 \end{vmatrix} \\
&= \frac{1}{R^{12}}\left\{(R^2-2u^2)(R^2-2v^2)(R^2-2w^2)-8u^2v^2w^2-8u^2v^2w^2\right. \\
&\qquad \left. -4u^2w^2(R^2-2v^2)-4v^2w^2(R^2-2u^2)-4u^2v^2(R^2-2w^2)\right\} \\
&= \frac{1}{R^{12}}\left\{R^6-2(u^2+v^2+w^2)R^4+4R^2(u^2v^2+u^2w^2+v^2w^2)-24u^2v^2w^2\right. \\
&\qquad \left. -4R^2(u^2w^2+v^2w^2+u^2v^2)+8u^2v^2w^2+8u^2v^2w^2+8u^2v^2w^2\right\} \\
&= \frac{1}{R^{12}}\{-R^6\} = -\frac{1}{R^6}.
\end{aligned}$$

Finally, we get by the transformation theorem and a consideration of a volume that

$$\begin{aligned}
I &= \int_A \frac{1}{r^6}\,d\Omega = \int_D R^6\cdot\frac{1}{R^6}\,d\omega = \int_D d\omega = \text{vol}(D) \\
&= \frac{1}{8}\cdot\frac{4\pi}{3}-\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{\pi}{6}-\frac{1}{48},
\end{aligned}$$

in which the slicing method is latently applied.